

GENERALIZED HARISH-CHANDRA DESCENT, GELFAND PAIRS AND AN ARCHIMEDEAN ANALOG OF JACQUET-RALLIS' THEOREM

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ABSTRACT. In the first part of the paper we generalize a descent technique due to Harish-Chandra to the case of a reductive group acting on a smooth affine variety both defined over an arbitrary local field F of characteristic zero. Our main tool is the Luna Slice Theorem.

In the second part of the paper we apply this technique to symmetric pairs. In particular we prove that the pairs $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ and $(\mathrm{GL}_n(E), \mathrm{GL}_n(F))$ are Gelfand pairs for any local field F and its quadratic extension E . In the non-Archimedean case, the first result was proven earlier by Jacquet and Rallis and the second by Flicker.

We also prove that any conjugation invariant distribution on $\mathrm{GL}_n(F)$ is invariant with respect to transposition. For non-Archimedean F the latter is a classical theorem of Gelfand and Kazhdan.

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1. INTRODUCTION

Harish-Chandra developed a technique based on Jordan decomposition that allows to reduce certain statements on conjugation invariant distributions on a reductive group to the set of unipotent elements, provided that the statement is known for certain subgroups (see e.g. [HC99]).

In this paper we generalize an aspect of this technique to the setting of a reductive group acting on a smooth affine algebraic variety, using the Luna Slice Theorem. Our technique is oriented towards proving Gelfand property for pairs of reductive groups.

Our approach is uniform for all local fields of characteristic zero – both Archimedean and non-Archimedean.

1.1. Main results.

The core of this paper is Theorem 3.1.1:

Theorem. *Let a reductive group G act on a smooth affine variety X , both defined over a local field F of characteristic zero. Let χ be a character of $G(F)$.*

Suppose that for any $x \in X(F)$ with closed orbit there are no non-zero distributions on the normal space at x to the orbit $G(F)x$ which are $(G(F)_x, \chi)$ -equivariant, where G_x denotes the stabilizer of x .

Then there are no non-zero $(G(F), \chi)$ -equivariant distributions on $X(F)$.

In fact, a stronger version based on this theorem is given in Corollary 3.2.2. This stronger version is based on an inductive argument. It shows that it is enough to prove that there are no non-zero equivariant distributions on the normal space to the orbit $G(F)x$ at x under the assumption that all such distributions are supported in a certain closed subset which is the analog of the nilpotent cone.

We apply this stronger version to problems of the following type. Let a reductive group G act on a smooth affine variety X , and τ be an involution of X which normalizes the image of G in $\mathrm{Aut}(X)$. We want to check whether any $G(F)$ -invariant distribution on $X(F)$ is also τ -invariant. Evidently, there is the following necessary condition on τ :

(*) Any closed orbit in $X(F)$ is τ -invariant.

In some cases this condition is also sufficient. In these cases we call the action of G on X *tame*.

This is a weakening of the property called "*density*" in [RR96]. However, it is sufficient for the purpose of proving Gelfand property for pairs of reductive groups.

In §6 we give criteria for tameness of actions. In particular, we introduce the notion of "*special*" action in order to show that certain actions are tame (see Theorem 6.0.5 and Proposition 7.3.5). Also, in many cases one can verify that an action is special using purely algebraic-geometric means.

In the second part of the paper we restrict our attention to the case of symmetric pairs. We transfer the terminology on actions to terminology on symmetric pairs. For example, we call a symmetric pair (G, H) *tame* if the action of $H \times H$ on G is tame.

In addition we introduce the notion of a "*regular*" symmetric pair (see Definition 7.4.2), which also helps to prove Gelfand property. Namely, we prove Theorem 7.4.5.

Theorem. *Let G be a reductive group defined over a local field F and let θ be an involution of G . Let $H := G^\theta$ and let σ be the anti-involution defined by $\sigma(g) := \theta(g^{-1})$. Consider the symmetric pair (G, H) .*

Suppose that all its "descendants" (including itself, see Definition 7.2.2) are regular. Suppose also that any closed $H(F)$ -double coset in $G(F)$ is σ -invariant.

Then every bi- $H(F)$ -invariant distribution on $G(F)$ is σ -invariant. In particular, by Gelfand-Kazhdan criterion, the pair (G, H) is a Gelfand pair (see §8).

Also, we formulate an algebraic-geometric criterion for regularity of a pair (Proposition 7.3.7). We sum up the various properties of symmetric pairs and their interrelations in a diagram in Appendix E.

As an application and illustration of our methods we prove in §7.7 that the pair $(\mathrm{GL}_{n+k}, \mathrm{GL}_n \times \mathrm{GL}_k)$ is a Gelfand pair by proving that it is regular, along with its descendants. In the non-Archimedean case this was proven in [JR96] and our proof is along the same lines. Our technique enabled us to streamline some of the computations in the proof of [JR96] and to extend it to the Archimedean case.

We also prove (in §7.6) that the pair $(G(E), G(F))$ is tame for any reductive group G over F and a quadratic field extension E/F . This implies that the pair $(\mathrm{GL}_n(E), \mathrm{GL}_n(F))$ is a Gelfand pair. In the non-Archimedean case this was proven in [Fli91]. Also we prove that the adjoint action of a reductive group on itself is tame. This is a generalization of a classical theorem by Gelfand and Kazhdan, see [GK75].

In general, we conjecture that any symmetric pair is regular. This would imply the van Dijk conjecture:

Conjecture (van Dijk). *Any symmetric pair (G, H) over \mathbb{C} such that G/H is connected is a Gelfand pair.*

1.2. Related work.

This paper was inspired by the paper [JR96] by Jacquet and Rallis where they prove that the pair $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ is a Gelfand pair for any non-Archimedean local field F of characteristic zero. Our aim was to see to what extent their techniques generalize.

Another generalization of Harish-Chandra descent using the Luna Slice Theorem has been carried out in the non-Archimedean case in [RR96]. In that paper Rader and Rallis investigated spherical characters of H -distinguished representations of G for symmetric pairs (G, H) and checked the validity of what they call the "*density principle*" for rank one symmetric pairs. They found out that the principle usually holds, but also found counterexamples.

In [vD86], van-Dijk investigated rank one symmetric pairs in the Archimedean case and classified the Gelfand pairs among them. In [BvD94], van-Dijk and Bosman studied the non-Archimedean case and obtained results for most rank one symmetric pairs. We hope that the second part of our paper will enhance the understanding of this question for symmetric pairs of higher rank.

1.3. Structure of the paper.

In §2 we introduce notation and terminology which allows us to speak uniformly about spaces of points of smooth algebraic varieties over Archimedean and non-Archimedean local fields, and equivariant distributions on those spaces.

In §§2.3 we formulate a version of the Luna Slice Theorem for points over local fields (Theorem 2.3.17). In §§2.5 we formulate results on equivariant distributions and equivariant Schwartz distributions. Most of those results are borrowed from [BZ76], [Ber84], [Bar03] and [AGS08], and the rest are proven in Appendix B.

In §3 we formulate and prove the Generalized Harish-Chandra Descent Theorem and its stronger version.

§4 is of interest only in the Archimedean case. In that section we prove that in the cases at hand if there are no equivariant Schwartz distributions then there are no equivariant distributions at all. Schwartz distributions are discussed in Appendix B.

In §5 we formulate a homogeneity Theorem which helps us to check the conditions of the Generalized Harish-Chandra Descent Theorem. In the non-Archimedean case this theorem had been proved earlier (see e.g. [JR96], [RS07] or [AGRS07]). We provide the proof for the Archimedean case in Appendix C.

In §6 we introduce the notion of tame actions and provide tameness criteria.

In §7 we apply our tools to symmetric pairs. In §7.3 we provide criteria for tameness of a symmetric pair. In §7.4 we introduce the notion of a regular symmetric pair and prove Theorem 7.4.5 alluded to above. In §7.5 we discuss conjectures about the regularity and the Gelfand property of symmetric pairs. In §7.6 we prove that certain symmetric pairs are tame. In §7.7 we prove that the pair $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ is regular.

In §8 we recall basic facts on Gelfand pairs and their connections to invariant distributions. We also prove that the pairs $(\mathrm{GL}_{n+k}(F), \mathrm{GL}_n(F) \times \mathrm{GL}_k(F))$ and $(\mathrm{GL}_n(E), \mathrm{GL}_n(F))$ are Gelfand pairs for any local field F and its quadratic extension E .

We start Appendix A by discussing different versions of the Inverse Function Theorem for local fields. Then we prove a version of the Luna Slice Theorem for points over local fields (Theorem 2.3.17). For Archimedean F this was done by Luna himself in [Lun75].

Appendices B and C are of interest only in the Archimedean case.

In Appendix B we discuss Schwartz distributions on Nash manifolds. We prove Frobenius reciprocity for them and construct the pullback of a Schwartz distribution under a Nash submersion. Also we prove that G -invariant distributions which are (Nashly) compactly supported modulo G are Schwartz distributions.

In Appendix C we prove the Archimedean version of the Homogeneity Theorem discussed in §5.

In Appendix D we formulate and prove a version of Bernstein's Localization Principle (Theorem 4.0.1). This appendix is of interest only for Archimedean F since for l -spaces a more general version of this principle had been proven in [Ber84]. This appendix is used in §4.

In [AGS09] we formulated Localization Principle in the setting of differential geometry. Admittedly, we currently do not have a proof of this principle in such a general setting. However, in Appendix D we present a proof in the case of a reductive group G acting on a smooth affine variety X . This generality is sufficiently wide for all applications we encountered up to now, including the one considered in [AGS09].

Finally, in Appendix E we present a diagram that illustrates the interrelations of various properties of symmetric pairs.

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Part 1. Generalized Harish-Chandra descent

2. PRELIMINARIES AND NOTATION

2.1. Conventions.

- Henceforth we fix a local field F of characteristic zero. All the algebraic varieties and algebraic groups that we will consider will be defined over F .
- For a group G acting on a set X we denote by X^G the set of fixed points of X . Also, for an element $x \in X$ we denote by G_x the stabilizer of x .

- By a reductive group we mean a (non-necessarily connected) algebraic reductive group.
- We consider an algebraic variety X defined over F as an algebraic variety over \overline{F} together with action of the Galois group $\text{Gal}(\overline{F}/F)$. On X we only consider the Zariski topology. On $X(F)$ we only consider the analytic (Hausdorff) topology. We treat finite-dimensional linear spaces defined over F as algebraic varieties.
- The tangent space of a manifold (algebraic, analytic, etc.) X at x will be denoted by $T_x X$.
- Usually we will use the letters X, Y, Z, Δ to denote algebraic varieties and the letters G, H to denote reductive groups. We will usually use the letters $V, W, U, K, M, N, C, O, S, T$ to denote analytic spaces (such as F -points of algebraic varieties) and the letter K to denote analytic groups. Also we will use the letters L, V, W to denote vector spaces of all kinds.

2.2. Categorical quotient.

Definition 2.2.1. *Let an algebraic group G act on an algebraic variety X . A pair consisting of an algebraic variety Y and a G -invariant morphism $\pi : X \rightarrow Y$ is called **the quotient of X by the action of G** if for any pair (π', Y') , there exists a unique morphism $\phi : Y \rightarrow Y'$ such that $\pi' = \phi \circ \pi$. Clearly, if such pair exists it is unique up to a canonical isomorphism. We will denote it by $(\pi_X, X/G)$.*

Theorem 2.2.2 (cf. [Dre00]). *Let a reductive group G act on an affine variety X . Then the quotient X/G exists, and every fiber of the quotient map π_X contains a unique closed orbit. In fact, $X/G := \text{Spec } \mathcal{O}(X)^G$.*

2.3. Algebraic geometry over local fields.

2.3.1. Analytic manifolds.

In this paper we consider distributions over l -spaces, smooth manifolds and Nash manifolds. l -spaces are locally compact totally disconnected topological spaces and Nash manifolds are semi-algebraic smooth manifolds.

For basic facts on l -spaces and distributions over them we refer the reader to [BZ76, §1].

For basic facts on Nash manifolds and Schwartz functions and distributions over them see Appendix B and [AG08a]. In this paper we consider only separated Nash manifolds.

We now introduce notation and terminology which allows a uniform treatment of the Archimedean and the non-Archimedean cases.

We will use the notion of an analytic manifold over a local field (see e.g. [Ser64, Part II, Chapter III]). When we say "**analytic manifold**" we always mean analytic manifold over some local field. Note that an analytic manifold over a non-Archimedean field is in particular an l -space and an analytic manifold over an Archimedean field is in particular a smooth manifold.

Definition 2.3.1. *A **B-analytic manifold** is either an analytic manifold over a non-Archimedean local field, or a Nash manifold.*

Remark 2.3.2. *If X is a smooth algebraic variety, then $X(F)$ is a B-analytic manifold and $(T_x X)(F) = T_x(X(F))$.*

Notation 2.3.3. *Let M be an analytic manifold and S be an analytic submanifold. We denote by $N_S^M := (T_M|_Y)/T_S$ the **normal bundle to S in M** . The **conormal bundle** is defined by $CN_S^M := (N_S^M)^*$. Denote by $\text{Sym}^k(CN_S^M)$ the k -th symmetric power of the conormal bundle. For a point $y \in S$ we denote by $N_{S,y}^M$ the normal space to S in M at the point y and by $CN_{S,y}^M$ the conormal space.*

2.3.2. G -orbits on X and $G(F)$ -orbits on $X(F)$.

Lemma 2.3.4 (see Appendix A.1). *Let G be an algebraic group and let $H \subset G$ be a closed subgroup. Then $G(F)/H(F)$ is open and closed in $(G/H)(F)$.*

Corollary 2.3.5. *Let an algebraic group G act on an algebraic variety X . Let $x \in X(F)$. Then*

$$N_{Gx,x}^X(F) \cong N_{G(F)x,x}^{X(F)}$$

Proposition 2.3.6. *Let an algebraic group G act on an algebraic variety X . Suppose that $S \subset X(F)$ is a non-empty closed $G(F)$ -invariant subset. Then S contains a closed orbit.*

Proof. The proof is by Noetherian induction on X . Choose $x \in S$. Consider $Z := \overline{Gx} - Gx$.

If $Z(F) \cap S$ is empty then $Gx(F) \cap S$ is closed and hence $G(F)x \cap S$ is closed by Lemma 2.3.4. Therefore $G(F)x$ is closed.

If $Z(F) \cap S$ is non-empty then $Z(F) \cap S$ contains a closed orbit by the induction assumption. \square

Corollary 2.3.7. *Let an algebraic group G act on an algebraic variety X . Let U be an open $G(F)$ -invariant subset of $X(F)$. Suppose that U contains all closed $G(F)$ -orbits. Then $U = X(F)$.*

Theorem 2.3.8 ([RR96], §2 fact A, pages 108-109). *Let a reductive group G act on an affine variety X . Let $x \in X(F)$. Then the following are equivalent:*

- (i) $G(F)x \subset X(F)$ is closed (in the analytic topology).
- (ii) $Gx \subset X$ is closed (in the Zariski topology).

Definition 2.3.9. *Let a reductive group G act on an affine variety X . We call an element $x \in X$ G -semisimple if its orbit Gx is closed.*

In particular, in the case where G acts on itself by conjugation, the notion of G -semisimplicity coincides with the usual one.

Notation 2.3.10. *Let V be an F -rational finite-dimensional representation of a reductive group G . We set*

$$Q_G(V) := Q(V) := (V/V^G)(F).$$

Since G is reductive, there is a canonical embedding $Q(V) \hookrightarrow V(F)$. Let $\pi : V(F) \rightarrow (V/G)(F)$ be the natural map. We set

$$\Gamma_G(V) := \Gamma(V) := \pi^{-1}(\pi(0)).$$

Note that $\Gamma(V) \subset Q(V)$. We also set

$$R_G(V) := R(V) := Q(V) - \Gamma(V).$$

Notation 2.3.11. *Let a reductive group G act on an affine variety X . For a G -semisimple element $x \in X(F)$ we set*

$$S_x := \{y \in X(F) \mid \overline{G(F)y} \ni x\}.$$

Lemma 2.3.12. *Let V be an F -rational finite-dimensional representation of a reductive group G . Then $\Gamma(V) = S_0$.*

This lemma follows from [RR96, fact A on page 108] for non-Archimedean F and [Brk71, Theorem 5.2 on page 459] for Archimedean F .

Example 2.3.13. *Let a reductive group G act on its Lie algebra \mathfrak{g} by the adjoint action. Then $\Gamma(\mathfrak{g})$ is the set of nilpotent elements of \mathfrak{g} .*

Proposition 2.3.14. *Let a reductive group G act on an affine variety X . Let $x, z \in X(F)$ be G -semisimple elements which do not lie in the same orbit of $G(F)$. Then there exist disjoint $G(F)$ -invariant open neighborhoods U_x of x and U_z of z .*

For the proof of this Proposition see [Lun75] for Archimedean F and [RR96, fact B on page 109] for non-Archimedean F .

Corollary 2.3.15. *Let a reductive group G act on an affine variety X . Suppose that $x \in X(F)$ is a G -semisimple element. Then the set S_x is closed.*

Proof. Let $y \in \overline{S_x}$. By Proposition 2.3.6, $\overline{G(F)y}$ contains a closed orbit $G(F)z$. If $G(F)z = G(F)x$ then $y \in S_x$. Otherwise, choose disjoint open G -invariant neighborhoods U_z of z and U_x of x . Since $z \in \overline{G(F)y}$, U_z intersects $G(F)y$ and hence contains y . Since $y \in \overline{S_x}$, this means that U_z intersects S_x . Let $t \in U_z \cap S_x$. Since U_z is $G(F)$ -invariant, $G(F)t \subset U_z$. By the definition of S_x , $x \in \overline{G(F)t}$ and hence $x \in \overline{U_z}$. Hence U_z intersects U_x – contradiction! \square

2.3.3. Analytic Luna slices.

Definition 2.3.16. Let a reductive group G act on an affine variety X . Let $\pi : X(F) \rightarrow (X/G)(F)$ be the natural map. An open subset $U \subset X(F)$ is called **saturated** if there exists an open subset $V \subset (X/G)(F)$ such that $U = \pi^{-1}(V)$.

We will use the following corollary of the Luna Slice Theorem:

Theorem 2.3.17 (see Appendix A.2). Let a reductive group G act on a smooth affine variety X . Let $x \in X(F)$ be G -semisimple. Consider the natural action of the stabilizer G_x on the normal space $N_{G_x, x}^X$. Then there exist

- (i) an open $G(F)$ -invariant B -analytic neighborhood U of $G(F)x$ in $X(F)$ with a G -equivariant B -analytic retract $p : U \rightarrow G(F)x$ and
- (ii) a G_x -equivariant B -analytic embedding $\psi : p^{-1}(x) \hookrightarrow N_{G_x, x}^X(F)$ with an open saturated image such that $\psi(x) = 0$.

Definition 2.3.18. In the notation of the previous theorem, denote $S := p^{-1}(x)$ and $N := N_{G_x, x}^X(F)$. We call the quintuple (U, p, ψ, S, N) an **analytic Luna slice at x** .

Corollary 2.3.19. In the notation of the previous theorem, let $y \in p^{-1}(x)$. Denote $z := \psi(y)$. Then

- (i) $(G(F)_x)_z = G(F)_y$
- (ii) $N_{G(F)_y, y}^{X(F)} \cong N_{G(F)_x z, z}^N$ as $G(F)_y$ -spaces
- (iii) y is G -semisimple if and only if z is G_x -semisimple.

2.4. Vector systems.¹

In this subsection we introduce the term "vector system". This term allows to formulate statements in wider generality.

Definition 2.4.1. For an analytic manifold M we define the notions of a **vector system** and a **B -vector system** over it.

For a smooth manifold M , a vector system over M is a pair (E, B) where B is a smooth locally trivial fibration over M and E is a smooth (finite-dimensional) vector bundle over B .

For a Nash manifold M , a B -vector system over M is a pair (E, B) where B is a Nash fibration over M and E is a Nash (finite-dimensional) vector bundle over B .

For an l -space M , a vector system over M (or a B -vector system over M) is a sheaf of complex linear spaces.

In particular, in the case where M is a point, a vector system over M is either a \mathbb{C} -vector space if F is non-Archimedean, or a smooth manifold together with a vector bundle in the case where F is Archimedean. The simplest example of a vector system over a manifold M is given by the following.

Definition 2.4.2. Let \mathcal{V} be a vector system over a point pt . Let M be an analytic manifold. A **constant vector system with fiber \mathcal{V}** is the pullback of \mathcal{V} with respect to the map $M \rightarrow pt$. We denote it by \mathcal{V}_M .

2.5. Distributions.

Definition 2.5.1. Let M be an analytic manifold over F . We define $C_c^\infty(M)$ in the following way.

If F is non-Archimedean then $C_c^\infty(M)$ is the space of locally constant compactly supported complex valued functions on M . We do not consider any topology on $C_c^\infty(M)$.

If F is Archimedean then $C_c^\infty(M)$ is the space of smooth compactly supported complex valued functions on M , endowed with the standard topology.

For any analytic manifold M , we define the space of distributions $\mathcal{D}(M)$ by $\mathcal{D}(M) := C_c^\infty(M)^*$. We consider the weak topology on it.

¹Subsection 2.4 and in particular the notion of "vector system" along with the results at the end of §§3.1 and §§3.2 are not essential for the rest of the paper. They are merely included for future reference.

Definition 2.5.2. Let M be a B -analytic manifold. We define $\mathcal{S}(M)$ in the following way.

If M is an analytic manifold over non-Archimedean field, $\mathcal{S}(M) := C_c^\infty(M)$.

If M is a Nash manifold, $\mathcal{S}(M)$ is the space of Schwartz functions on M , namely smooth functions which are rapidly decreasing together with all their derivatives. See [AG08a] for the precise definition. We consider $\mathcal{S}(M)$ as a Fréchet space.

For any B -analytic manifold M , we define the space of **Schwartz distributions** $\mathcal{S}^*(M)$ by $\mathcal{S}^*(M) := \mathcal{S}(M)^*$. Clearly, $\mathcal{S}(M)^*$ is naturally embedded into $\mathcal{D}(M)$.

Notation 2.5.3. Let M be an analytic manifold. For a distribution $\xi \in \mathcal{D}(M)$ we denote by $\text{Supp}(\xi)$ the support of ξ .

For a closed subset $N \subset M$ we denote

$$\mathcal{D}_M(N) := \{\xi \in \mathcal{D}(M) \mid \text{Supp}(\xi) \subset N\}.$$

More generally, for a locally closed subset $N \subset M$ we denote

$$\mathcal{D}_M(N) := \mathcal{D}_{M \setminus (\overline{N} \setminus N)}(N).$$

Similarly if M is a B -analytic manifold and N is a locally closed subset we define $\mathcal{S}_M^*(N)$ in a similar vein.²

Definition 2.5.4. Let M be an analytic manifold over F and \mathcal{E} be a vector system over M . We define $C_c^\infty(M, \mathcal{E})$ in the following way.

If F is non-Archimedean then $C_c^\infty(M, \mathcal{E})$ is the space of compactly supported sections of \mathcal{E} .

If F is Archimedean and $\mathcal{E} = (E, B)$ where B is a fibration over M and E is a vector bundle over B , then $C_c^\infty(M, \mathcal{E})$ is the complexification of the space of smooth compactly supported sections of E over B .

If \mathcal{V} is a vector system over a point then we denote $C_c^\infty(M, \mathcal{V}) := C_c^\infty(M, \mathcal{V}_M)$.

We define $\mathcal{D}(M, \mathcal{E})$, $\mathcal{D}_M(N, \mathcal{E})$, $\mathcal{S}(M, \mathcal{E})$, $\mathcal{S}^*(M, \mathcal{E})$ and $\mathcal{S}_M^*(N, \mathcal{E})$ in the natural way.

Theorem 2.5.5. Let an l -group K act on an l -space M . Let $M = \bigcup_{i=0}^l M_i$ be a K -invariant stratification of M . Let χ be a character of K . Suppose that $\mathcal{S}^*(M_i)^{K, \chi} = 0$. Then $\mathcal{S}^*(M)^{K, \chi} = 0$.

This theorem is a direct corollary of [BZ76, Corollary 1.9].

For the proof of the next theorem see e.g. [AGS08, §B.2].

Theorem 2.5.6. Let a Nash group K act on a Nash manifold M . Let N be a locally closed subset. Let $N = \bigcup_{i=0}^l N_i$ be a Nash K -invariant stratification of N . Let χ be a character of K . Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq l$,

$$\mathcal{S}^*(N_i, \text{Sym}^k(CN_{N_i}^M))^{K, \chi} = 0.$$

Then $\mathcal{S}_M^*(N)^{K, \chi} = 0$.

Theorem 2.5.7 (Frobenius reciprocity). Let an analytic group K act on an analytic manifold M . Let N be an analytic manifold with a transitive action of K . Let $\phi : M \rightarrow N$ be a K -equivariant map.

Let $z \in N$ be a point and $M_z := \phi^{-1}(z)$ be its fiber. Let K_z be the stabilizer of z in K . Let Δ_K and Δ_{K_z} be the modular characters of K and K_z .

Let \mathcal{E} be a K -equivariant vector system over M . Then

(i) there exists a canonical isomorphism

$$\text{Fr} : \mathcal{D}(M_z, \mathcal{E}|_{M_z} \otimes \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1})^{K_z} \cong \mathcal{D}(M, \mathcal{E})^K.$$

In particular, Fr commutes with restrictions to open sets.

(ii) For B -analytic manifolds Fr maps $\mathcal{S}^*(M_z, \mathcal{E}|_{M_z} \otimes \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1})^{K_z}$ to $\mathcal{S}^*(M, \mathcal{E})^K$.

For the proof of (i) see [Ber84, §§1.5] and [BZ76, §§2.21 - 2.36] for the case of l -spaces and [AGS08, Theorem 4.2.3] or [Bar03] for smooth manifolds. For the proof of (ii) see Appendix B.

We will also use the following straightforward proposition.

²In the Archimedean case, locally closed is considered with respect to the restricted topology – cf. Appendix B.

Proposition 2.5.8. *Let K_i be analytic groups acting on analytic manifolds M_i for $i = 1 \dots n$. Let $\Omega_i \subset K_i$ be analytic subgroups. Let $\mathcal{E}_i \rightarrow M_i$ be K_i -equivariant vector systems. Suppose that*

$$\mathcal{D}(M_i, E_i)^{\Omega_i} = \mathcal{D}(M_i, E_i)^{K_i}$$

for all i . Then

$$\mathcal{D}\left(\prod M_i, \boxtimes E_i\right)^{\prod \Omega_i} = \mathcal{D}\left(\prod M_i, \boxtimes E_i\right)^{\prod K_i},$$

where \boxtimes denotes the external product.

Moreover, if Ω_i , K_i , M_i and \mathcal{E}_i are B -analytic then the analogous statement holds for Schwartz distributions.

For the proof see e.g. [AGS08, proof of Proposition 3.1.5].

3. GENERALIZED HARISH-CHANDRA DESCENT

3.1. Generalized Harish-Chandra descent.

In this subsection we will prove the following theorem.

Theorem 3.1.1. *Let a reductive group G act on a smooth affine variety X . Let χ be a character of $G(F)$. Suppose that for any G -semisimple $x \in X(F)$ we have*

$$\mathcal{D}(N_{G_x, x}^X(F))^{G(F), \chi} = 0.$$

Then

$$\mathcal{D}(X(F))^{G(F), \chi} = 0.$$

Remark 3.1.2. *In fact, the converse is also true. We will not prove it since we will not use it.*

For the proof of this theorem we will need the following lemma

Lemma 3.1.3. *Let a reductive group G act on a smooth affine variety X . Let χ be a character of $G(F)$. Let $U \subset X(F)$ be an open saturated subset. Suppose that $\mathcal{D}(X(F))^{G(F), \chi} = 0$. Then $\mathcal{D}(U)^{G(F), \chi} = 0$.*

Proof. Consider the quotient X/G . It is an affine algebraic variety. Embed it in an affine space \mathbb{A}^n . This defines a map $\pi : X(F) \rightarrow F^n$. Since U is saturated, there exists an open subset $V \subset (X/G)(F)$ such that $U = \pi^{-1}(V)$. Clearly there exists an open subset $V' \subset F^n$ such that $V' \cap (X/G)(F) = V$.

Let $\xi \in \mathcal{D}(U)^{G(F), \chi}$. Suppose that ξ is non-zero. Let $x \in \text{Supp} \xi$ and let $y := \pi(x)$. Let $g \in C_c^\infty(V')$ be such that $g(y) = 1$. Consider $\xi' \in \mathcal{D}(X(F))$ defined by $\xi'(f) := \xi(f \cdot (g \circ \pi))$. Clearly, $\text{Supp}(\xi') \subset U$ and hence we can interpret ξ' as an element in $\mathcal{D}(X(F))^{G(F), \chi}$. Therefore $\xi' = 0$. On the other hand, $x \in \text{Supp}(\xi')$. Contradiction. \square

Proof of Theorem 3.1.1. Let x be a G -semisimple element. Let $(U_x, p_x, \psi_x, S_x, N_x)$ be an analytic Luna slice at x .

Let $\xi' = \xi|_{U_x}$. Then $\xi' \in \mathcal{D}(U_x)^{G(F), \chi}$. By Frobenius reciprocity it corresponds to $\xi'' \in \mathcal{D}(S_x)^{G_x(F), \chi}$.

The distribution ξ'' corresponds to a distribution $\xi''' \in \mathcal{D}(\psi_x(S_x))^{G_x(F), \chi}$.

However, by the previous lemma the assumption implies that $\mathcal{D}(\psi_x(S_x))^{G_x(F), \chi} = 0$. Hence $\xi' = 0$.

Let $S \subset X(F)$ be the set of all G -semisimple points. Let $U = \bigcup_{x \in S} U_x$. We saw that $\xi|_U = 0$. On the other hand, U includes all the closed orbits, and hence by Corollary 2.3.7 $U = X$. \square

The following generalization of this theorem is proven in the same way.

Theorem 3.1.4. *Let a reductive group G act on a smooth affine variety X . Let $K \subset G(F)$ be an open subgroup and let χ be a character of K . Suppose that for any G -semisimple $x \in X(F)$ we have*

$$\mathcal{D}(N_{G_x, x}^X(F))^{K_x, \chi} = 0.$$

Then

$$\mathcal{D}(X(F))^{K, \chi} = 0.$$

Now we would like to formulate a slightly more general version of this theorem concerning K -equivariant vector systems. ³

³Subsection 2.4 and in particular the notion of "vector system" along with the results at the end of §§3.1 and §§3.2 are not essential for the rest of the paper. They are merely included for future reference.

Definition 3.1.5. Let a reductive group G act on a smooth affine variety X . Let $K \subset G(F)$ be an open subgroup. Let \mathcal{E} be a K -equivariant vector system on $X(F)$. Let $x \in X(F)$ be G -semisimple. Let \mathcal{E}' be a K_x -equivariant vector system on $N_{G_x,x}^X(F)$. We say that \mathcal{E} and \mathcal{E}' are **compatible** if there exists an analytic Luna slice (U, p, ψ, S, N) such that $\mathcal{E}|_S = \psi^*(\mathcal{E}')$.

Note that if \mathcal{E} and \mathcal{E}' are constant with the same fiber then they are compatible.

The following theorem is proven in the same way as Theorem 3.1.1.

Theorem 3.1.6. Let a reductive group G act on a smooth affine variety X . Let $K \subset G(F)$ be an open subgroup and let \mathcal{E} be a K -equivariant vector system on $X(F)$. Suppose that for any G -semisimple $x \in X(F)$ there exists a K -equivariant vector system \mathcal{E}' on $N_{G_x,x}^X(F)$, compatible with \mathcal{E} such that

$$\mathcal{D}(N_{G_x,x}^X(F), \mathcal{E}')^{K_x} = 0.$$

Then

$$\mathcal{D}(X(F), \mathcal{E})^K = 0.$$

If \mathcal{E} and \mathcal{E}' are B-vector systems and K is an open B-analytic subgroup⁴ of $G(F)$ then the theorem also holds for Schwartz distributions. Namely, if $\mathcal{S}^*(N_{G_x,x}^X(F), \mathcal{E}')^{K_x} = 0$ for any G -semisimple $x \in X(F)$ then $\mathcal{S}^*(X(F), \mathcal{E})^K = 0$. The proof is the same.

3.2. A stronger version.

In this section we provide means to validate the conditions of Theorems 3.1.1, 3.1.4 and 3.1.6 based on an inductive argument.

More precisely, the goal of this section is to prove the following theorem.

Theorem 3.2.1. Let a reductive group G act on a smooth affine variety X . Let $K \subset G(F)$ be an open subgroup and let χ be a character of K . Suppose that for any G -semisimple $x \in X(F)$ such that

$$\mathcal{D}(R_{G_x}(N_{G_x,x}^X))^{K_x, \chi} = 0$$

we have

$$\mathcal{D}(Q_{G_x}(N_{G_x,x}^X))^{K_x, \chi} = 0.$$

Then for any G -semisimple $x \in X(F)$ we have

$$\mathcal{D}(N_{G_x,x}^X(F))^{K_x, \chi} = 0.$$

Together with Theorem 3.1.4, this theorem gives the following corollary.

Corollary 3.2.2. Let a reductive group G act on a smooth affine variety X . Let $K \subset G(F)$ be an open subgroup and let χ be a character of K . Suppose that for any G -semisimple $x \in X(F)$ such that

$$\mathcal{D}(R(N_{G_x,x}^X))^{K_x, \chi} = 0$$

we have

$$\mathcal{D}(Q(N_{G_x,x}^X))^{K_x, \chi} = 0.$$

Then $\mathcal{D}(X(F))^{K, \chi} = 0$.

From now till the end of the section we fix G , X , K and χ . Let us introduce several definitions and notation.

Notation 3.2.3. Denote

- $T \subset X(F)$ the set of all G -semisimple points.
- For $x, y \in T$ we say that $x > y$ if $G_x \not\supseteq G_y$.
- $T_0 := \{x \in T \mid \mathcal{D}(Q(N_{G_x,x}^X))^{K_x, \chi} = 0\} = \{x \in T \mid \mathcal{D}((N_{G_x,x}^X))^{K_x, \chi} = 0\}$.

⁴In fact, any open subgroup of a B-analytic group is B-analytic.

Proof of Theorem 3.2.1. We have to show that $T = T_0$. Assume the contrary.

Note that every chain in T with respect to our ordering has a minimum. Hence by Zorn's lemma every non-empty set in T has a minimal element. Let x be a minimal element of $T - T_0$. To get a contradiction, it is enough to show that $\mathcal{D}(R(N_{G_x,x}^X))^{K_x,\chi} = 0$.

Denote $R := R(N_{G_x,x}^X)$. By Theorem 3.1.4, it is enough to show that for any $y \in R$ we have

$$\mathcal{D}(N_{G(F)_xy,y}^R)^{(K_x)_y,\chi} = 0.$$

Let (U, p, ψ, S, N) be an analytic Luna slice at x .

Since $\psi(S)$ is open and contains 0, we can assume, upon replacing y by λy for some $\lambda \in F^\times$, that $y \in \psi(S)$. Let $z \in S$ be such that $\psi(z) = y$. By Corollary 2.3.19, $G(F)_z = (G(F)_x)_y \subsetneq G(F)_x$ and $N_{G(F)_xy,y}^R \cong N_{Gz,z}^X(F)$. Hence $(K_x)_y = K_z$ and therefore

$$\mathcal{D}(N_{G(F)_xy,y}^R)^{(K_x)_y,\chi} \cong \mathcal{D}(N_{Gz,z}^X(F))^{K_z,\chi}.$$

However $z < x$ and hence $z \in T_0$ which means that $\mathcal{D}(N_{Gz,z}^X(F))^{K_z,\chi} = 0$. \square

Remark 3.2.4. *One can rewrite this proof such that it will use Zorn's lemma for finite sets only, which does not depend on the axiom of choice.*

Remark 3.2.5. *As before, Theorem 3.2.1 and Corollary 3.2.2 also hold for Schwartz distributions, with a similar proof.*

Again, we can formulate a more general version of Corollary 3.2.2 concerning vector systems.⁵

Theorem 3.2.6. *Let a reductive group G act on a smooth affine variety X . Let $K \subset G(F)$ be an open subgroup and let \mathcal{E} be a K -equivariant vector system on $X(F)$.*

Suppose that for any G -semisimple $x \in X(F)$ satisfying
 (*) *for any $K_x \times F^\times$ -equivariant vector system \mathcal{E}' on $R(N_{G_x,x}^X)$ (where F^\times acts by homothety) compatible with \mathcal{E} we have $\mathcal{D}(R(N_{G_x,x}^X), \mathcal{E}')^{K_x} = 0$,*

the following holds

(**) *there exists a $K_x \times F^\times$ -equivariant vector system \mathcal{E}' on $Q(N_{G_x,x}^X)$ compatible with \mathcal{E} such that*

$$\mathcal{D}(Q(N_{G_x,x}^X), \mathcal{E}')^{K_x} = 0.$$

Then $\mathcal{D}(X(F), \mathcal{E})^K = 0$.

The proof is the same as the proof of Theorem 3.2.1 using the following lemma which follows from the definitions.

Lemma 3.2.7. *Let a reductive group G act on a smooth affine variety X . Let $K \subset G(F)$ be an open subgroup and let \mathcal{E} be a K -equivariant vector system on $X(F)$. Let $x \in X(F)$ be G -semisimple. Let (U, p, ψ, S, N) be an analytic Luna slice at x .*

Let \mathcal{E}' be a K_x -equivariant vector system on N compatible with \mathcal{E} . Let $y \in S$ be G -semisimple, and let $z := \psi(y)$. Let \mathcal{E}'' be a $(K_x)_z$ -equivariant vector system on $N_{G_xz,z}^N$ compatible with \mathcal{E}' . Consider the isomorphism $N_{G_xz,z}^N(F) \cong N_{Gy,y}^X(F)$ and let \mathcal{E}''' be the corresponding K_y -equivariant vector system on $N_{Gy,y}^X(F)$.

Then \mathcal{E}''' is compatible with \mathcal{E} .

Again, if \mathcal{E} and \mathcal{E}' are B-vector systems then the theorem holds also for Schwartz distributions.

⁵Subsection 2.4 and in particular, the notion of "vector system" along with the results at the end of §§3.1 and §§3.2 are not essential for the rest of the paper. They are merely included for future reference.

4. DISTRIBUTIONS VERSUS SCHWARTZ DISTRIBUTIONS

In this section F is Archimedean. The tools developed in the previous section enable us to prove the following version of the Localization Principle.

Theorem 4.0.1 (Localization Principle). *Let a reductive group G act on a smooth algebraic variety X . Let Y be an algebraic variety and $\phi : X \rightarrow Y$ be an affine algebraic G -invariant map. Let χ be a character of $G(F)$. Suppose that for any $y \in Y(F)$ we have $\mathcal{D}_{X(F)}((\phi^{-1}(y))(F))^{G(F),\chi} = 0$. Then $\mathcal{D}(X(F))^{G(F),\chi} = 0$.*

For the proof see Appendix D.

In this section we use this theorem to show that if there are no $G(F)$ -equivariant Schwartz distributions on $X(F)$ then there are no $G(F)$ -equivariant distributions on $X(F)$.

Theorem 4.0.2. *Let a reductive group G act on a smooth affine variety X . Let V be a finite-dimensional algebraic representation of $G(F)$. Suppose that*

$$\mathcal{S}^*(X(F), V)^{G(F)} = 0.$$

Then

$$\mathcal{D}(X(F), V)^{G(F)} = 0.$$

For the proof we will need the following definition and theorem.

Definition 4.0.3.

(i) *Let a topological group K act on a topological space M . We call a closed K -invariant subset $C \subset M$ **compact modulo K** if there exists a compact subset $C' \subset M$ such that $C \subset KC'$.*

(ii) *Let a Nash group K act on a Nash manifold M . We call a closed K -invariant subset $C \subset M$ **Nashly compact modulo K** if there exist a compact subset $C' \subset M$ and semi-algebraic closed subset $Z \subset M$ such that $C \subset Z \subset KC'$.*

Remark 4.0.4. *Let a reductive group G act on a smooth affine variety X . Let $K := G(F)$ and $M := X(F)$. Then it is easy to see that the notions of compact modulo K and Nashly compact modulo K coincide.*

Theorem 4.0.5. *Let a Nash group K act on a Nash manifold M . Let E be a K -equivariant Nash bundle over M . Let $\xi \in \mathcal{D}(M, E)^K$ be such that $\text{Supp}(\xi)$ is Nashly compact modulo K . Then $\xi \in \mathcal{S}^*(M, E)^K$.*

The statement and the idea of the proof of this theorem are due to J. Bernstein. For the proof see Appendix B.4.

Proof of Theorem 4.0.2. Fix any $y \in (X/G)(F)$ and denote $M := \pi_X^{-1}(y)(F)$.

By the Localization Principle (Theorem 4.0.1 and Remark D.0.4), it is enough to prove that

$$\mathcal{S}_{X(F)}^*(M, V)^{G(F)} = \mathcal{D}_{X(F)}(M, V)^{G(F)}.$$

Choose $\xi \in \mathcal{D}_{X(F)}(M, V)^{G(F)}$. M has a unique closed stable G -orbit and hence a finite number of closed $G(F)$ -orbits. By Theorem 4.0.5, it is enough to show that M is Nashly compact modulo $G(F)$. Clearly M is semi-algebraic. Choose representatives x_i of the closed $G(F)$ -orbits in M . Choose compact neighborhoods C_i of x_i . Let $C' := \bigcup C_i$. By Corollary 2.3.7, $G(F)C' \supset M$. \square

5. APPLICATIONS OF FOURIER TRANSFORM AND THE WEIL REPRESENTATION

Let G be a reductive group and V be a finite-dimensional F -rational representation of G . Let χ be a character of $G(F)$. In this section we provide some tools to verify that $\mathcal{S}^*(Q(V))^{G(F),\chi} = 0$ provided that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$.

5.1. Preliminaries.

For this subsection let B be a non-degenerate bilinear form on a finite-dimensional vector space V over F . We also fix an additive character κ of F . If F is Archimedean we take $\kappa(x) := e^{2\pi i \operatorname{Re}(x)}$.

Notation 5.1.1. We identify V and V^* via B and endow V with the self-dual Haar measure with respect to ψ . Denote by $\mathcal{F}_B : \mathcal{S}^*(V) \rightarrow \mathcal{S}^*(V)$ the Fourier transform. For any B -analytic manifold M over F we also denote by $\mathcal{F}_B : \mathcal{S}^*(M \times V) \rightarrow \mathcal{S}^*(M \times V)$ the partial Fourier transform.

Notation 5.1.2. Consider the homothety action of F^\times on V given by $\rho(\lambda)v := \lambda^{-1}v$. It gives rise to an action ρ of F^\times on $\mathcal{S}^*(V)$.

Let $|\cdot|$ denote the normalized absolute value. Recall that for $F = \mathbb{R}$, $|\lambda|$ is equal to the classical absolute value but for $F = \mathbb{C}$, $|\lambda| = (\operatorname{Re} \lambda)^2 + (\operatorname{Im} \lambda)^2$.

Notation 5.1.3. We denote by $\gamma(B)$ the Weil constant. For its definition see e.g. [Gel76, §2.3] for non-Archimedean F and [RS78, §1] for Archimedean F .

For any $t \in F^\times$ denote $\delta_B(t) = \gamma(B)/\gamma(tB)$.

Note that $\gamma(B)$ is an 8-th root of unity and if $\dim V$ is odd and $F \neq \mathbb{C}$ then δ_B is **not** a multiplicative character.

Notation 5.1.4. We denote

$$Z(B) := \{x \in V \mid B(x, x) = 0\}.$$

Theorem 5.1.5 (non-Archimedean homogeneity). *Suppose that F is **non-Archimedean**. Let M be a B -analytic manifold over F . Let $\xi \in \mathcal{S}_{V \times M}^*(Z(B) \times M)$ be such that $\mathcal{F}_B(\xi) \in \mathcal{S}_{V \times M}^*(Z(B) \times M)$. Then for any $t \in F^\times$, we have $\rho(t)\xi = \delta_B(t)|t|^{\dim V/2}\xi$ and $\xi = \gamma(B)^{-1}\mathcal{F}_B(\xi)$. In particular, if $\dim V$ is odd then $\xi = 0$.*

For the proof see e.g. [RS07, §§8.1] or [JR96, §§3.1].

For the Archimedean version of this theorem we will need the following definition.

Definition 5.1.6. Let M be a B -analytic manifold over F . We say that a distribution $\xi \in \mathcal{S}^*(V \times M)$ is **adapted to B** if either

- (i) for any $t \in F^\times$ we have $\rho(t)\xi = \delta(t)|t|^{\dim V/2}\xi$ and ξ is proportional to $\mathcal{F}_B\xi$ or
- (ii) F is Archimedean and for any $t \in F^\times$ we have $\rho(t)\xi = \delta(t)t|t|^{\dim V/2}\xi$.

Note that if $\dim V$ is odd and $F \neq \mathbb{C}$ then every B -adapted distribution is zero.

Theorem 5.1.7 (Archimedean homogeneity). *Let M be a Nash manifold. Let $L \subset \mathcal{S}_{V \times M}^*(Z(B) \times M)$ be a non-zero subspace such that for all $\xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B \cdot \xi \in L$ (here B is viewed as a quadratic function).*

Then there exists a non-zero distribution $\xi \in L$ which is adapted to B .

For Archimedean F we prove this theorem in Appendix C. For non-Archimedean F it follows from Theorem 5.1.5.

We will also use the following trivial observation.

Lemma 5.1.8. *Let a B -analytic group K act linearly on V and preserving B . Let M be a B -analytic K -manifold over F . Let $\xi \in \mathcal{S}^*(V \times M)$ be a K -invariant distribution. Then $\mathcal{F}_B(\xi)$ is also K -invariant.*

5.2. Applications.

The following two theorems easily follow from the results of the previous subsection.

Theorem 5.2.1. *Suppose that F is non-Archimedean. Let G be a reductive group. Let V be a finite-dimensional F -rational representation of G . Let χ be character of $G(F)$. Suppose that $\mathcal{S}^*(R(V))^{G(F), \chi} = 0$. Let $V = V_1 \oplus V_2$ be a G -invariant decomposition of V . Let B be a G -invariant symmetric non-degenerate bilinear form on V_1 . Consider the action ρ of F^\times on V by homothety on V_1 .*

Then any $\xi \in \mathcal{S}^(Q(V))^{G(F), \chi}$ satisfies $\rho(t)\xi = \delta_B(t)|t|^{\dim V_1/2}\xi$ and $\xi = \gamma(B)\mathcal{F}_B\xi$. In particular, if $\dim V_1$ is odd then $\xi = 0$.*

Theorem 5.2.2. *Let G be a reductive group. Let V be a finite-dimensional F -rational representation of G . Let χ be character of $G(F)$. Suppose that $\mathcal{S}^*(R(V))^{G(F),\chi} = 0$. Let $Q(V) = W \oplus (\bigoplus_{i=1}^k V_i)$ be a G -invariant decomposition of $Q(V)$. Let B_i be G -invariant symmetric non-degenerate bilinear forms on V_i . Suppose that any $\xi \in \mathcal{S}_{Q(V)}^*(\Gamma(V))^{G(F),\chi}$ which is adapted to each B_i is zero.*

Then $\mathcal{S}^(Q(V))^{G(F),\chi} = 0$.*

Remark 5.2.3. *One can easily generalize Theorems 5.2.2 and 5.2.1 to the case of constant vector systems.*

6. TAME ACTIONS

In this section we consider problems of the following type. A reductive group G acts on a smooth affine variety X , and τ is an automorphism of X which normalizes the image of G in $\text{Aut}(X)$. We want to check whether any $G(F)$ -invariant Schwartz distribution on $X(F)$ is also τ -invariant.

Definition 6.0.1. *Let π be an action of a reductive group G on a smooth affine variety X . We say that an algebraic automorphism τ of X is **G -admissible** if*

- (i) τ normalizes $\pi(G(F))$ and $\tau^2 \in \pi(G(F))$.
- (ii) For any closed $G(F)$ -orbit $O \subset X(F)$, we have $\tau(O) = O$.

Proposition 6.0.2. *Let π be an action of a reductive group G on a smooth affine variety X . Let τ be a G -admissible automorphism of X . Let $K := \pi(G(F))$ and let \tilde{K} be the group generated by $\pi(G(F))$ and τ . Let $x \in X(F)$ be a point with closed $G(F)$ -orbit. Let $\tau' \in \tilde{K}_x - K_x$. Then $d\tau'|_{N_{G_x,x}^X}$ is G_x -admissible.*

Proof. Let \tilde{G} denote the group generated by $\pi(G)$ and τ . We check that the two properties of G_x -admissibility hold for $d\tau'|_{N_{G_x,x}^X}$. The first one is obvious. For the second, let $y \in N_{G_x,x}^X(F)$ be an element with closed G_x -orbit. Let $y' = d\tau'(y)$. We have to show that there exists $g \in G_x(F)$ such that $gy = y'$. Let (U, p, ψ, S, N) be an analytic Luna slice at x with respect to the action of \tilde{G} . We can assume that there exists $z \in S$ such that $y = \psi(z)$. Let $z' = \tau'(z)$. By Corollary 2.3.19, z is G -semisimple. Since τ is admissible, this implies that there exists $g \in G(F)$ such that $gz = z'$. Clearly, $g \in G_x(F)$ and $gy = y'$. \square

Definition 6.0.3. *We call an action of a reductive group G on a smooth affine variety X **tame** if for any G -admissible $\tau : X \rightarrow X$, we have $\mathcal{S}^*(X(F))^{G(F)} \subset \mathcal{S}^*(X(F))^\tau$.*

Definition 6.0.4. *We call an F -rational representation V of a reductive group G **linearly tame** if for any G -admissible linear map $\tau : V \rightarrow V$, we have $\mathcal{S}^*(V(F))^{G(F)} \subset \mathcal{S}^*(V(F))^\tau$.*

*We call a representation **weakly linearly tame** if for any G -admissible linear map $\tau : V \rightarrow V$, such that $\mathcal{S}^*(R(V))^{G(F)} \subset \mathcal{S}^*(R(V))^\tau$ we have $\mathcal{S}^*(Q(V))^{G(F)} \subset \mathcal{S}^*(Q(V))^\tau$.*

Theorem 6.0.5. *Let a reductive group G act on a smooth affine variety X . Suppose that for any G -semisimple $x \in X(F)$, the action of G_x on $N_{G_x,x}^X$ is weakly linearly tame. Then the action of G on X is tame.*

The proof is rather straightforward except for one minor complication: the group of automorphisms of $X(F)$ generated by the action of $G(F)$ is not necessarily a group of F -points of any algebraic group.

Proof. Let $\tau : X \rightarrow X$ be an admissible automorphism.

Let $\tilde{G} \subset \text{Aut}(X)$ be the algebraic group generated by the actions of G and τ . Let $K \subset \text{Aut}(X(F))$ be the B -analytic group generated by the action of $G(F)$. Let $\tilde{K} \subset \text{Aut}(X(F))$ be the B -analytic group generated by the actions of G and τ . Note that $\tilde{K} \subset \tilde{G}(F)$ is an open subgroup of finite index. Note that for any $x \in X(F)$, x is \tilde{G} -semisimple if and only if it is G -semisimple. If $K = \tilde{K}$ we are done, so we will assume $K \neq \tilde{K}$. Let χ be the character of \tilde{K} defined by $\chi(K) = \{1\}$, $\chi(\tilde{K} - K) = \{-1\}$.

It is enough to prove that $\mathcal{S}^*(X)^{\tilde{K},\chi} = 0$. By Generalized Harish-Chandra Descent (Corollary 3.2.2) it is enough to prove that for any G -semisimple $x \in X$ such that

$$\mathcal{S}^*(R(N_{G_x,x}^X))^{\tilde{K}_x,\chi} = 0$$

we have

$$\mathcal{S}^*(Q(N_{G_x,x}^X))^{\tilde{K}_x,\chi} = 0.$$

Choose any automorphism $\tau' \in \tilde{K}_x - K_x$. Note that τ' and K_x generate \tilde{K}_x . Denote

$$\eta := d\tau'|_{N_{G_x,x}^X(F)}.$$

By Proposition 6.0.2, η is G_x -admissible. Note that

$$\mathcal{S}^*(R(N_{G_x,x}^X))^{K_x} = \mathcal{S}^*(R(N_{G_x,x}^X))^{G(F)_x} \text{ and } \mathcal{S}^*(Q(N_{G_x,x}^X))^{K_x} = \mathcal{S}^*(Q(N_{G_x,x}^X))^{G(F)_x}.$$

Hence we have

$$\mathcal{S}^*(R(N_{G_x,x}^X))^{G(F)_x} \subset \mathcal{S}^*(R(N_{G_x,x}^X))^\eta.$$

Since the action of G_x is weakly linearly tame, this implies that

$$\mathcal{S}^*(Q(N_{G_x,x}^X))^{G(F)_x} \subset \mathcal{S}^*(Q(N_{G_x,x}^X))^\eta$$

and therefore $\mathcal{S}^*(Q(N_{G_x,x}^X))^{\tilde{K}_x,\chi} = 0$. □

Definition 6.0.6. We call an F -rational representation V of a reductive group G **special** if there is no non-zero $\xi \in \mathcal{S}_{Q(V)}^*(\Gamma(V))^{G(F)}$ such that for any G -invariant decomposition $Q(V) = W_1 \oplus W_2$ and any two G -invariant symmetric non-degenerate bilinear forms B_i on W_i the Fourier transforms $\mathcal{F}_{B_i}(\xi)$ are also supported in $\Gamma(V)$.

Proposition 6.0.7. Every special representation V of a reductive group G is weakly linearly tame.

The proposition follows immediately from the following lemma.

Lemma 6.0.8. Let V be an F -rational representation of a reductive group G . Let τ be an admissible linear automorphism of V . Let $V = W_1 \oplus W_2$ be a G -invariant decomposition of V and B_i be G -invariant symmetric non-degenerate bilinear forms on W_i . Then W_i and B_i are also τ -invariant.

This lemma follows in turn from the following one.

Lemma 6.0.9. Let V be an F -rational representation of a reductive group G . Let τ be an admissible automorphism of V . Then $\mathcal{O}(V)^G \subset \mathcal{O}(V)^\tau$.

Proof. Consider the projection $\pi : V \rightarrow V/G$. We have to show that τ acts trivially on V/G and let $x \in \pi(V(F))$. Let $X := \pi^{-1}(x)$. By Proposition 2.3.6 $G(F)$ has a closed orbit in $X(F)$. The automorphism τ preserves this orbit and hence preserves x . Thus τ acts trivially on $\pi(V(F))$, which is Zariski dense in V/G . Hence τ acts trivially on V/G . □

Now we introduce a criterion that allows to prove that a representation is special. It follows immediately from Theorem 5.1.7.

Lemma 6.0.10. Let V be an F -rational representation of a reductive group G . Let $Q(V) = \bigoplus W_i$ be a G -invariant decomposition. Let B_i be symmetric non-degenerate G -invariant bilinear forms on W_i . Suppose that any $\xi \in \mathcal{S}_{Q(V)}^*(\Gamma(V))^{G(F)}$ which is adapted to all B_i is zero. Then V is special.

Part 2. Symmetric and Gelfand pairs

7. SYMMETRIC PAIRS

In this section we apply our tools to symmetric pairs. We introduce several properties of symmetric pairs and discuss their interrelations. In Appendix E we present a diagram that illustrates the most important ones.

7.1. Preliminaries and notation.

Definition 7.1.1. A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^\theta$. We call a symmetric pair **connected** if G/H is connected.

For a symmetric pair (G, H, θ) we define an antiinvolution $\sigma : G \rightarrow G$ by

$$\sigma(g) := \theta(g^{-1}),$$

denote $\mathfrak{g} := \text{Lie } G$, $\mathfrak{h} := \text{Lie } H$. Let θ and σ act on \mathfrak{g} by their differentials and denote

$$\mathfrak{g}^\sigma := \{a \in \mathfrak{g} \mid \sigma(a) = a\} = \{a \in \mathfrak{g} \mid \theta(a) = -a\}.$$

Note that H acts on \mathfrak{g}^σ by the adjoint action. Denote also

$$G^\sigma := \{g \in G \mid \sigma(g) = g\}$$

and define a **symmetrization map** $s : G \rightarrow G^\sigma$ by

$$s(g) := g\sigma(g).$$

We will consider the action of $H \times H$ on G by left and right translation and the conjugation action of H on G^σ .

Definition 7.1.2. Let (G_1, H_1, θ_1) and (G_2, H_2, θ_2) be symmetric pairs. We define their **product** to be the symmetric pair $(G_1 \times G_2, H_1 \times H_2, \theta_1 \times \theta_2)$.

Theorem 7.1.3. For any connected symmetric pair (G, H, θ) we have $\mathcal{O}(G)^{H \times H} \subset \mathcal{O}(G)^\sigma$.

Proof. Consider the multiplication map $H \times G^\sigma \rightarrow G$. It is étale at 1×1 and hence its image HG^σ contains an open neighborhood of 1 in G . Hence the image of HG^σ in G/H is dense. Thus $HG^\sigma H$ is dense in G . Clearly $\mathcal{O}(HG^\sigma H)^{H \times H} \subset \mathcal{O}(HG^\sigma H)^\sigma$ and hence $\mathcal{O}(G)^{H \times H} \subset \mathcal{O}(G)^\sigma$. \square

Corollary 7.1.4. For any connected symmetric pair (G, H, θ) and any closed $H \times H$ orbit $\Delta \subset G$, we have $\sigma(\Delta) = \Delta$.

Proof. Denote $\Upsilon := H \times H$. Consider the action of the 2-element group $(1, \tau)$ on Υ given by $\tau(h_1, h_2) := (\theta(h_2), \theta(h_1))$. This defines the semi-direct product $\tilde{\Upsilon} := (1, \tau) \ltimes \Upsilon$. Extend the two-sided action of Υ to $\tilde{\Upsilon}$ by the antiinvolution σ . Note that the previous theorem implies that $G/\Upsilon = G/\tilde{\Upsilon}$. Let Δ be a closed Υ -orbit. Let $\tilde{\Delta} := \Delta \cup \sigma(\Delta)$. Let $a := \pi_G(\tilde{\Delta}) \subset G/\tilde{\Upsilon}$. Clearly, a consists of one point. On the other hand, $G/\tilde{\Upsilon} = G/\Upsilon$ and hence $\pi_G^{-1}(a)$ contains a unique closed G -orbit. Therefore $\Delta = \tilde{\Delta} = \sigma(\Delta)$. \square

Corollary 7.1.5. Let (G, H, θ) be a connected symmetric pair. Let $g \in G(F)$ be $H \times H$ -semisimple. Suppose that the Galois cohomology $H^1(F, (H \times H)_g)$ is trivial. Then $\sigma(g) \in H(F)gH(F)$.

For example, if $(H \times H)_g$ is a product of general linear groups over some field extensions then $H^1(F, (H \times H)_g)$ is trivial.

Definition 7.1.6. A symmetric pair (G, H, θ) is called **good** if for any closed $H(F) \times H(F)$ orbit $O \subset G(F)$, we have $\sigma(O) = O$.

Corollary 7.1.7. Any connected symmetric pair over \mathbb{C} is good.

Definition 7.1.8. A symmetric pair (G, H, θ) is called a **GK-pair** if

$$\mathcal{S}^*(G(F))^{H(F) \times H(F)} \subset \mathcal{S}^*(G(F))^\sigma.$$

We will see later in §8 that GK-pairs satisfy a Gelfand pair property that we call GP2 (see Definition 8.1.2 and Theorem 8.1.5). Clearly every GK-pair is good and we conjecture that the converse is also true. We will discuss it in more detail in §§7.5.

Lemma 7.1.9. Let (G, H, θ) be a symmetric pair. Then there exists a G -invariant θ -invariant non-degenerate symmetric bilinear form B on \mathfrak{g} . In particular, $\mathfrak{g} = \mathfrak{g}^\sigma \oplus \mathfrak{h}$ is an orthogonal direct sum with respect to B .

Proof.

Step 1. Proof for semisimple \mathfrak{g} .

Let B be the Killing form on \mathfrak{g} . Since it is non-degenerate, it is enough to show that \mathfrak{h} is orthogonal to \mathfrak{g}^σ . Let $A \in \mathfrak{h}$ and $C \in \mathfrak{g}^\sigma$. We have to show $\text{Tr}(\text{ad}(A)\text{ad}(C)) = 0$. This follows from the fact that $\text{ad}(A)\text{ad}(C)(\mathfrak{h}) \subset \mathfrak{g}^\sigma$ and $\text{ad}(A)\text{ad}(C)(\mathfrak{g}^\sigma) \subset \mathfrak{h}$.

Step 2. Proof in the general case.

Let $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ such that \mathfrak{g}' is semisimple and \mathfrak{z} is the center. It is easy to see that this decomposition is invariant under $\text{Aut}(\mathfrak{g})$ and hence θ -invariant. Now the proposition easily follows from the previous case. \square

Remark 7.1.10. *Let (G, H, θ) be a symmetric pair. Let $\mathcal{U}(G)$ be the set of unipotent elements in $G(F)$ and $\mathcal{N}(\mathfrak{g})$ the set of nilpotent elements in $\mathfrak{g}(F)$. Then the exponent map $\exp : \mathcal{N}(\mathfrak{g}) \rightarrow \mathcal{U}(G)$ is σ -equivariant and intertwines the adjoint action with conjugation.*

Lemma 7.1.11. *Let (G, H, θ) be a symmetric pair. Let $x \in \mathfrak{g}^\sigma$ be a nilpotent element. Then there exists a group homomorphism $\phi : \text{SL}_2 \rightarrow G$ such that*

$$d\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = x, \quad d\phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) \in \mathfrak{g}^\sigma \quad \text{and} \quad \phi\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \in H.$$

In particular $0 \in \overline{\text{Ad}(H)(x)}$.

This lemma was essentially proven for $F = \mathbb{C}$ in [KR73]. The same proof works for any F and we repeat it here for the convenience of the reader.

Proof. By the Jacobson-Morozov Theorem (see [Jac62, Chapter III, Theorems 17 and 10]) we can complete x to an \mathfrak{sl}_2 -triple (x_-, s, x) . Let $s' := \frac{s+\theta(s)}{2}$. It satisfies $[s', x] = 2x$ and lies in the ideal $[x, \mathfrak{g}]$ and hence by the Morozov Lemma (see [Jac62, Chapter III, Lemma 7]), x and s' can be completed to an \mathfrak{sl}_2 triple (x_-, s', x) . Let $x'_- := \frac{x-\theta(x_-)}{2}$. Note that (x'_-, s', x) is also an \mathfrak{sl}_2 -triple. Exponentiating this \mathfrak{sl}_2 -triple to a map $\text{SL}_2 \rightarrow G$ we get the required homomorphism. \square

Notation 7.1.12. *In the notation of the previous lemma we denote*

$$D_t(x) := \phi\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \in H \quad \text{and} \quad d(x) := d\phi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \in \mathfrak{h}.$$

These elements depend on the choice of ϕ . However, whenever we use this notation, nothing will depend on their choice.

7.2. Descendants of symmetric pairs. Recall that for a symmetric pair (G, H, θ) we consider the $H \times H$ action on G by left and right translation and the conjugation action of H on G^σ .

Proposition 7.2.1. *Let (G, H, θ) be a symmetric pair. Let $g \in G(F)$ be $H \times H$ -semisimple. Let $x = s(g)$. Then*

- (i) x is semisimple (both as an element of G and with respect to the H -action).
- (ii) $H_x \cong (H \times H)_g$ and $(\mathfrak{g}_x)^\sigma \cong N_{H \times H, g}^G$ as H_x -spaces.

Proof.

(i) Since the symmetrization map is closed, it is clear that the H -orbit of x is closed. This means that x is semisimple with respect to the H -action. Now we have to show that x is semisimple as an element of G . Let $x = x_s x_u$ be the Jordan decomposition of x . The uniqueness of the Jordan decomposition implies that both x_u and x_s belong to G^σ . To show that $x_u = 1$ it is enough to show that $\overline{\text{Ad}(H)(x)} \ni x_s$. We will do that in several steps.

Step 1. Proof for the case when $x_s = 1$.

It follows immediately from Remark 7.1.10 and Lemma 7.1.11.

Step 2. Proof for the case when $x_s \in Z(G)$.

This case follows from Step 1 since conjugation acts trivially on $Z(G)$.

Step 3. Proof in the general case.

Note that $x \in G_{x_s}$ and G_{x_s} is θ -invariant. The statement follows from Step 2 for the group G_{x_s} .

(ii) The symmetrization map gives rise to an isomorphism $(H \times H)_g \cong H_x$. Let us now show that $(\mathfrak{g}_x)^\sigma \cong N_{H_g H, g}^G$. First of all, $N_{H_g H, g}^G \cong \mathfrak{g}/(\mathfrak{h} + \text{Ad}(g)\mathfrak{h})$. Let θ' be the involution of G defined by $\theta'(y) = x\theta(y)x^{-1}$. Note that $\text{Ad}(g)\mathfrak{h} = \mathfrak{g}^{\theta'}$. Fix a non-degenerate G -invariant symmetric bilinear form B on \mathfrak{g} as in Lemma 7.1.9. Note that B is also θ' -invariant and hence

$$(\text{Ad}(g)\mathfrak{h})^\perp = \{a \in \mathfrak{g} \mid \theta'(a) = -a\}.$$

Now

$$N_{H_g H, g}^G \cong (\mathfrak{h} + \text{Ad}(g)\mathfrak{h})^\perp = \mathfrak{h}^\perp \cap \text{Ad}(g)\mathfrak{h}^\perp = \{a \in \mathfrak{g} \mid \theta(a) = \theta'(a) = -a\} = (\mathfrak{g}_x)^\sigma.$$

□

It is easy to see that the isomorphism $N_{H_g H, g}^G \cong (\mathfrak{g}_x)^\sigma$ is independent of the choice of B .

Definition 7.2.2. *In the notation of the previous proposition we will say that the pair $(G_x, H_x, \theta|_{G_x})$ is a **descendant** of (G, H, θ) .*

7.3. Tame symmetric pairs.

Definition 7.3.1. *We call a symmetric pair (G, H, θ)*

(i) **tame** if the action of $H \times H$ on G is tame.

(ii) **linearly tame** if the action of H on \mathfrak{g}^σ is linearly tame.

(iii) **weakly linearly tame** if the action of H on \mathfrak{g}^σ is weakly linearly tame.

Remark 7.3.2. *Evidently, any good tame symmetric pair is a GK-pair.*

The following theorem is a direct corollary of Theorem 6.0.5.

Theorem 7.3.3. *Let (G, H, θ) be a symmetric pair. Suppose that all its descendants (including itself) are weakly linearly tame. Then (G, H, θ) is tame and linearly tame.*

Definition 7.3.4. *We call a symmetric pair (G, H, θ) **special** if \mathfrak{g}^σ is a special representation of H (see Definition 6.0.6).*

The following proposition follows immediately from Proposition 6.0.7.

Proposition 7.3.5. *Any special symmetric pair is weakly linearly tame.*

Using Lemma 7.1.9 it is easy to prove the following proposition.

Proposition 7.3.6. *A product of special symmetric pairs is special.*

Now we would like to give a criterion of speciality for symmetric pairs. Recall the notation $d(x)$ of 7.1.12.

Proposition 7.3.7 (Speciality criterion). *Let (G, H, θ) be a symmetric pair. Suppose that for any nilpotent $x \in \mathfrak{g}^\sigma$ either*

(i) $\text{Tr}(\text{ad}(d(x))|_{\mathfrak{h}_x}) < \dim \mathfrak{g}^\sigma$ or

(ii) F is non-Archimedean and $\text{Tr}(\text{ad}(d(x))|_{\mathfrak{h}_x}) \neq \dim \mathfrak{g}^\sigma$.

Then the pair (G, H, θ) is special.

For the proof we will need the following auxiliary results.

Lemma 7.3.8. *Let (G, H, θ) be a symmetric pair. Then $\Gamma(\mathfrak{g}^\sigma)$ is the set of all nilpotent elements in $Q(\mathfrak{g}^\sigma)$.*

This lemma is a direct corollary from Lemma 7.1.11.

Lemma 7.3.9. *Let (G, H, θ) be a symmetric pair. Let $x \in \mathfrak{g}^\sigma$ be a nilpotent element. Then all the eigenvalues of $\text{ad}(d(x))|_{\mathfrak{g}^\sigma/[x, \mathfrak{h}]}$ are non-positive integers.*

This lemma follows from the existence of a natural surjection $\mathfrak{g}/[x, \mathfrak{g}] \twoheadrightarrow \mathfrak{g}^\sigma/[x, \mathfrak{h}]$ (given by the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^\sigma$)

using the following straightforward lemma.

Lemma 7.3.10. *Let V be a representation of an \mathfrak{sl}_2 triple (e, h, f) . Then all the eigenvalues of $h|_{V/e(V)}$ are non-positive integers.*

Now we are ready to prove the speciality criterion.

Proof of Proposition 7.3.7. We will give a proof in the case where F is Archimedean. The case of non-Archimedean F is done in the same way but with less complications.

By Lemma 6.0.10 and the definition of adapted it is enough to prove

$$\mathcal{S}_{Q(\mathfrak{g}^\sigma)}^*(\Gamma(\mathfrak{g}^\sigma))^{H(F) \times F^\times, (1, \chi)} = 0$$

for any character χ of F^\times of the form $\chi(\lambda) = u(\lambda)|\lambda|^{\dim \mathfrak{g}^\sigma/2}$ or $\chi(\lambda) = u(\lambda)|\lambda|^{\dim \mathfrak{g}^\sigma/2+1}$, where u is some unitary character.

The set $\Gamma(\mathfrak{g}^\sigma)$ has a finite number of $H(F)$ -orbits (it follows from Lemma 7.3.8 and the introduction of [KR73]). Hence it is enough to show that for any $x \in \Gamma(\mathfrak{g}^\sigma)$ we have

$$\mathcal{S}^*(\text{Ad}(H(F))x, \text{Sym}^k(\text{CN}_{\text{Ad}(H(F))x}^{\mathfrak{g}^\sigma}))^{H(F) \times F^\times, (1, \chi)} = 0 \text{ for any } k.$$

Let $K := \{(D_t(x), t^2) | t \in F^\times\} \subset (H(F) \times F^\times)_x$.

Note that

$$\Delta_{(H(F) \times F^\times)_x}((D_t(x), t^2)) = |\det(\text{Ad}(D_t(x))|_{\mathfrak{g}_x^\sigma})| = |t|^{\text{Tr}(\text{ad}(d(x))|_{\mathfrak{h}_x})}.$$

By Lemma 7.3.9 the eigenvalues of the action of $(D_t(x), t^2)$ on $(\text{Sym}^k(\mathfrak{g}^\sigma/[x, \mathfrak{h}]))$ are of the form t^l where l is a non-positive integer.

Now by Frobenius reciprocity (Theorem 2.5.7) we have

$$\begin{aligned} \mathcal{S}^* \left((H(F))x, \text{Sym}^k(\text{CN}_{\text{Ad}(H(F))x}^{\mathfrak{g}^\sigma}) \right)^{H(F) \times F^\times, (1, \chi)} &= \\ = \mathcal{S}^* \left(\{x\}, \text{Sym}^k(\text{CN}_{\text{Ad}(H(F))x, x}^{\mathfrak{g}^\sigma}) \otimes \Delta_{H(F) \times F^\times} |_{(H(F) \times F^\times)_x} \cdot \Delta_{(H(F) \times F^\times)_x}^{-1} \otimes (1, \chi) \right)^{(H(F) \times F^\times)_x} &= \\ = \left(\text{Sym}^k(\mathfrak{g}^\sigma/[x, \mathfrak{h}]) \otimes \Delta_{(H(F) \times F^\times)_x} \otimes (1, \chi)^{-1} \otimes_{\mathbb{R}} \mathbb{C} \right)^{(H(F) \times F^\times)_x} \subset & \\ \subset \left(\text{Sym}^k(\mathfrak{g}^\sigma/[x, \mathfrak{h}]) \otimes \Delta_{(H(F) \times F^\times)_x} \otimes (1, \chi)^{-1} \otimes_{\mathbb{R}} \mathbb{C} \right)^K & \end{aligned}$$

which is zero since all the absolute values of the eigenvalues of the action of any $(D_t(x), t^2) \in K$ on

$$\text{Sym}^k(\mathfrak{g}^\sigma/[x, \mathfrak{h}]) \otimes \Delta_{(H(F) \times F^\times)_x} \otimes (1, \chi)^{-1}$$

are of the form $|t|^l$ where $l < 0$. □

7.4. Regular symmetric pairs.

In this subsection we will formulate a property which is weaker than weakly linearly tame but still enables us to prove the GK property for good pairs.

Definition 7.4.1. *Let (G, H, θ) be a symmetric pair. We call an element $g \in G(F)$ **admissible** if*

- (i) $\text{Ad}(g)$ commutes with θ (or, equivalently, $s(g) \in Z(G)$) and
- (ii) $\text{Ad}(g)|_{\mathfrak{g}^\sigma}$ is H -admissible.

Definition 7.4.2. *We call a symmetric pair (G, H, θ) **regular** if for any admissible $g \in G(F)$ such that $\mathcal{S}^*(R(\mathfrak{g}^\sigma))^{H(F)} \subset \mathcal{S}^*(R(\mathfrak{g}^\sigma))^{\text{Ad}(g)}$ we have*

$$\mathcal{S}^*(Q(\mathfrak{g}^\sigma))^{H(F)} \subset \mathcal{S}^*(Q(\mathfrak{g}^\sigma))^{\text{Ad}(g)}.$$

Remark 7.4.3. *Clearly, every weakly linearly tame pair is regular.*

Proposition 7.4.4. *Let (G_1, H_1, θ_1) and (G_2, H_2, θ_2) be regular symmetric pairs. Then their product $(G_1 \times G_2, H_1 \times H_2, \theta_1 \times \theta_2)$ is also a regular pair.*

Proof. This follows from Proposition 2.5.8, since a product of admissible elements is admissible, and $R(\mathfrak{g}_1^{\sigma_2}) \times R(\mathfrak{g}_2^{\sigma_2})$ is an open saturated subset of $R((\mathfrak{g}_1 \times \mathfrak{g}_2)^{\sigma_1 \times \sigma_2})$. □

The goal of this subsection is to prove the following theorem.

Theorem 7.4.5. *Let (G, H, θ) be a good symmetric pair such that all its descendants are regular. Then it is a GK-pair.*

We will need several definitions and lemmas.

Definition 7.4.6. *Let (G, H, θ) be a symmetric pair. An element $g \in G$ is called **normal** if g commutes with $\sigma(g)$.*

Note that if g is normal then

$$g\sigma(g)^{-1} = \sigma(g)^{-1}g \in H.$$

The following lemma is straightforward.

Lemma 7.4.7. *Let (G, H, θ) be a symmetric pair. Then any σ -invariant $H(F) \times H(F)$ -orbit in $G(F)$ contains a normal element.*

Proof.

Let $g' \in O$. We know that $\sigma(g') = h_1g'h_2$ where $h_1, h_2 \in H(F)$. Let $g := g'h_1$. Then

$$\begin{aligned} \sigma(g)g &= h_1^{-1}\sigma(g')g'h_1 = h_1^{-1}\sigma(g')\sigma(\sigma(g'))h_1 = \\ &= h_1^{-1}h_1g'h_2\sigma(h_1g'h_2)h_1 = g'\sigma(g') = g'h_1h_1^{-1}\sigma(g') = g\sigma(g). \end{aligned}$$

Thus g in O is normal. □

Notation 7.4.8. *Let (G, H, θ) be a symmetric pair. We denote*

$$\widetilde{H \times H} := H \times H \rtimes \{1, \sigma\}$$

where

$$\sigma \cdot (h_1, h_2) = (\theta(h_2), \theta(h_1)) \cdot \sigma.$$

The two-sided action of $H \times H$ on G is extended to an action of $\widetilde{H \times H}$ in the natural way. We denote by χ the character of $\widetilde{H \times H}$ defined by

$$\chi(\widetilde{H \times H} - H \times H) = \{-1\}, \quad \chi(H \times H) = \{1\}.$$

Proposition 7.4.9. *Let (G, H, θ) be a good symmetric pair. Let $O \subset G(F)$ be a closed $H(F) \times H(F)$ -orbit.*

Then for any $g \in O$ there exist $\tau \in (\widetilde{H \times H})_g(F) - (H \times H)_g(F)$ and $g' \in G_{s(g)}(F)$ such that $\text{Ad}(g')$ commutes with θ on $G_{s(g)}$ and the action of τ on $N_{O,g}^G$ corresponds via the isomorphism given by Proposition 7.2.1 to the adjoint action of g' on $\mathfrak{g}_{s(g)}^\sigma$.

Proof. Clearly, if the statement holds for some $g \in O$ then it holds for all $g \in O$.

Let $g \in O$ be a normal element. Let $h := g\sigma(g)^{-1}$. Recall that $h \in H(F)$ and $gh = hg = \sigma(g)$. Let $\tau := (h^{-1}, 1) \cdot \sigma$. Evidently, $\tau \in (\widetilde{H \times H})_g(F) - (H \times H)_g(F)$. Consider $d\tau_g : T_g G \rightarrow T_g G$. It corresponds via the identification $dg : \mathfrak{g} \cong T_g G$ to some $A : \mathfrak{g} \rightarrow \mathfrak{g}$. Clearly, $A = da$ where $a : G \rightarrow G$ is defined by $a(\alpha) = g^{-1}h^{-1}\sigma(g\alpha)$. However, $g^{-1}h^{-1}\sigma(g\alpha) = \theta(g)\sigma(\alpha)\theta(g)^{-1}$. Hence $A = \text{Ad}(\theta(g)) \circ \sigma$. By Lemma 7.1.9, there exists a non-degenerate G -invariant σ -invariant symmetric bilinear form B on \mathfrak{g} . By Theorem 7.1.3, A preserves B . Therefore τ corresponds to $A|_{\mathfrak{g}_{s(g)}^\sigma}$ via the isomorphism given by Proposition 7.2.1. However, σ is trivial on $\mathfrak{g}_{s(g)}^\sigma$ and hence $A|_{\mathfrak{g}_{s(g)}^\sigma} = \text{Ad}(\theta(g))|_{\mathfrak{g}_{s(g)}^\sigma}$. Since g is normal, $\theta(g) \in G_{s(g)}$. It is easy to see that $\text{Ad}(\theta(g))$ commutes with θ on $G_{s(g)}$. Hence we take $g' := \theta(g)$. □

Now we are ready to prove Theorem 7.4.5.

Proof of Theorem 7.4.5. We have to show that $\mathcal{S}^*(G(F))^{\widetilde{H \times H}, \chi} = 0$. By Theorem 3.2.2 it is enough to show that for any $H \times H$ -semisimple $x \in G(F)$ such that

$$\mathcal{D}(R(N_{HxH,x}^G))^{(H(F) \times H(F))_x, \chi} = 0$$

we have

$$\mathcal{D}(Q(N_{HxH,x}^G))^{(H(F) \times \widetilde{H(F)})_x \cdot \chi} = 0.$$

This follows immediately from the regularity of the pair (G_x, H_x) using the last proposition. \square

7.5. Conjectures.

Conjecture 1 (van Dijk). *If $F = \mathbb{C}$ then any connected symmetric pair is a Gelfand pair (GP3, see Definition 8.1.2 below).*

By Theorem 8.1.5 this would follow from the following stronger conjecture.

Conjecture 2. *If $F = \mathbb{C}$ then any connected symmetric pair is a GK-pair.*

By Corollary 7.1.7 this in turn would follow from the following more general conjecture.

Conjecture 3. *Every good symmetric pair is a GK-pair.*

which in turn follows (by Theorem 7.4.5) from the following one.

Conjecture 4. *Any symmetric pair is regular.*

Remark 7.5.1. *In the next two subsections we prove this conjecture for certain symmetric pairs. In subsequent works [AG08c, Say08a, AS08, Say08b, Aiz08] this conjecture was verified for most classical symmetric pairs and several exceptional ones.*

Remark 7.5.2. *An indirect evidence for this conjecture is that every GK-pair is regular. One can easily show this by analyzing a Luna slice for an orbit of an admissible element.*

Remark 7.5.3. *It is well known that if F is Archimedean, G is connected and H is compact then the pair (G, H, θ) is good, Gelfand (GP1, see Definition 8.1.2 below) and in fact also GK. See e.g. [Yak04].*

Remark 7.5.4. *In general, not every symmetric pair is good. For example, $(\mathrm{SL}_2(\mathbb{R}), T)$ where T is the split torus. Also, it is not a Gelfand pair (not even GP3, see Definition 8.1.2 below).*

Remark 7.5.5. *It seems unlikely that every symmetric pair is special. However, in the next two subsections we will prove that certain symmetric pairs are special.*

7.6. The pairs $(G \times G, \Delta G)$ and $(G_{E/F}, G)$ are tame.

Notation 7.6.1. *Let E be a quadratic extension of F . Let G be an algebraic group defined over F . We denote by $G_{E/F}$ the restriction of scalars from E to F of G viewed as a group over E . Thus, $G_{E/F}$ is an algebraic group defined over F and $G_{E/F}(F) = G(E)$.*

In this section we will prove the following theorem.

Theorem 7.6.2. *Let G be a reductive group.*

(i) *Consider the involution θ of $G \times G$ given by $\theta((g, h)) := (h, g)$. Its fixed points form the diagonal subgroup ΔG . Then the symmetric pair $(G \times G, \Delta G, \theta)$ is tame.*

(ii) *Let E be a quadratic extension of F . Consider the involution γ of $G_{E/F}$ given by the nontrivial element of $\mathrm{Gal}(E/F)$. Its fixed points form G . Then the symmetric pair $(G_{E/F}, G, \gamma)$ is tame.*

Corollary 7.6.3. *Let G be a reductive group. Then the adjoint action of G on itself is tame. In particular, every conjugation invariant distribution on $\mathrm{GL}_n(F)$ is transposition invariant⁶.*

For the proof of the theorem we will need the following straightforward lemma.

Lemma 7.6.4.

- (i) *Every descendant of $(G \times G, \Delta G, \theta)$ is of the form $(H \times H, \Delta H, \theta)$ for some reductive group H .*
- (ii) *Every descendant of $(G_{E/F}, G, \gamma)$ is of the form $(H_{E/F}, H, \gamma)$ for some reductive group H .*

Now in view of Theorem 7.4.5, Theorem 7.6.2 follows from the following theorem.

⁶In the non-Archimedean case, the latter is a classical result of Gelfand and Kazhdan, see [GK75].

Theorem 7.6.5. *The pairs $(G \times G, \Delta G, \theta)$ and $(G_{E/F}, G, \gamma)$ are special for any reductive group G .*

By the speciality criterion (Proposition 7.3.7) this theorem follows from the following lemma.

Lemma 7.6.6. *Let \mathfrak{g} be a semisimple Lie algebra. Let $\{e, h, f\} \subset \mathfrak{g}$ be an \mathfrak{sl}_2 triple. Then $\text{Tr}(\text{ad}(h)|_{\mathfrak{g}_e})$ is an integer smaller than $\dim \mathfrak{g}$.*

Proof. Consider \mathfrak{g} as a representation of \mathfrak{sl}_2 via the triple (e, h, f) . Decompose it into irreducible representations $\mathfrak{g} = \bigoplus V_i$. Let λ_i be the highest weights of V_i . Clearly

$$\text{Tr}(\text{ad}(h)|_{\mathfrak{g}_e}) = \sum \lambda_i \text{ while } \dim \mathfrak{g} = \sum (\lambda_i + 1).$$

□

7.7. The pair $(\text{GL}_{n+k}, \text{GL}_n \times \text{GL}_k)$ is a GK pair.

Notation 7.7.1. *We define an involution $\theta_{n,k} : \text{GL}_{n+k} \rightarrow \text{GL}_{n+k}$ by $\theta_{n,k}(x) = \varepsilon x \varepsilon$ where $\varepsilon = \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix}$. Note that $(\text{GL}_{n+k}, \text{GL}_n \times \text{GL}_k, \theta_{n,k})$ is a symmetric pair. If there is no ambiguity we will denote $\theta_{n,k}$ simply by θ .*

Theorem 7.7.2. *The pair $(\text{GL}_{n+k}, \text{GL}_n \times \text{GL}_k, \theta_{n,k})$ is a GK-pair.*

By Theorem 7.4.5 it is enough to prove that our pair is good and all its descendants are regular.

In §§§7.7.1 we compute the descendants of our pair and show that the pair is good.

In §§§7.7.2 we prove that all the descendants are regular.

7.7.1. The descendants of the pair $(\text{GL}_{n+k}, \text{GL}_n \times \text{GL}_k)$.

Theorem 7.7.3. *All the descendants of the pair $(\text{GL}_{n+k}, \text{GL}_n \times \text{GL}_k, \theta_{n,k})$ are products of pairs of the types*

- (i) $((\text{GL}_m)_{E/F} \times (\text{GL}_m)_{E/F}, \Delta(\text{GL}_m)_{E/F}, \theta)$ for some field extension E/F
- (ii) $((\text{GL}_m)_{E/F}, (\text{GL}_m)_{L/F}, \gamma)$ for some field extension L/F and its quadratic extension E/L
- (iii) $(\text{GL}_{m+l}, \text{GL}_m \times \text{GL}_l, \theta_{m,l})$.

Proof. Let $x \in \text{GL}_{n+k}^\sigma(F)$ be a semisimple element. We have to compute G_x and H_x . Since $x \in G^\sigma$, we have $\varepsilon x \varepsilon = x^{-1}$. Let $V = F^{n+k}$. Decompose $V := \bigoplus_{i=1}^s V_i$ such that the minimal polynomial of $x|_{V_i}$ is irreducible. Now $G_x(F)$ decomposes as a product of $\text{GL}_{E_i}(V_i)$, where E_i is the extension of F defined by the minimal polynomial of $x|_{V_i}$ and the E_i -vector space structure on V_i is given by x .

Clearly, ε permutes the V_i 's. Now we see that V is a direct sum of spaces of the following two types

- A. $W_1 \oplus W_2$ such that the minimal polynomials of $x|_{W_i}$ are irreducible and $\varepsilon(W_1) = W_2$.
- B. W such that the minimal polynomial of $x|_W$ is irreducible and $\varepsilon(W) = W$.

It is easy to see that in case A we get the symmetric pair (i).

In case B there are two possibilities: either $x = x^{-1}$ or $x \neq x^{-1}$. It is easy to see that these cases correspond to types (iii) and (ii) respectively. □

Corollary 7.7.4. *The pair $(\text{GL}_{n+k}, \text{GL}_n \times \text{GL}_k)$ is good.*

Proof. Theorem 7.7.3 implies that for any $(\text{GL}_n \times \text{GL}_k) \times (\text{GL}_n \times \text{GL}_k)$ -semisimple element $x \in \text{GL}_{n+k}(F)$, the stabilizer $((\text{GL}_n \times \text{GL}_k) \times (\text{GL}_n \times \text{GL}_k))_x$ is a product of groups of types $(\text{GL}_m)_{E/F}$ for some extensions E/F . Hence $H^1(F, ((\text{GL}_n \times \text{GL}_k) \times (\text{GL}_n \times \text{GL}_k))_x) = 0$ and hence by Corollary 7.1.5 the pair $(\text{GL}_{n+k}, \text{GL}_n \times \text{GL}_k)$ is good. □

7.7.2. All the descendants of the pair $(\text{GL}_{n+k}, \text{GL}_n \times \text{GL}_k)$ are regular.

Clearly, for any field extension E/F , if a pair (G, H, θ) is regular as a symmetric pair over E then the pair $(G_{E/F}, H_{E/F}, \theta)$ is regular. Therefore by Theorem 7.7.3 and Theorem 7.6.2 it is enough to prove that the pair $(\text{GL}_{n+k}, \text{GL}_n \times \text{GL}_k, \theta_{n,k})$ is regular as a symmetric pair over F .

In the case $n \neq k$ this follows from the definition since in this case the normalizer of $\text{GL}_n \times \text{GL}_k$ in GL_{n+k} is $\text{GL}_n \times \text{GL}_k$ and hence, any admissible $g \in \text{GL}_{n+k}$ lies in $\text{GL}_n \times \text{GL}_k$.

So we can assume $n = k > 0$. Hence by Proposition 7.3.7 it suffices to prove the following Key Lemma.

Lemma 7.7.5 (Key Lemma). ⁷ Let $x \in \mathfrak{gl}_{2n}^\sigma(F)$ be a nilpotent element and $d := d(x)$. Then

$$\mathrm{Tr}(\mathrm{ad}(d)|_{(\mathfrak{gl}_n(F) \times \mathfrak{gl}_n(F))_x}) < 2n^2.$$

We will need the following definition and lemmas.

Definition 7.7.6. We fix a grading on $\mathfrak{sl}_2(F)$ given by $h \in \mathfrak{sl}_2(F)_0$ and $e, f \in \mathfrak{sl}_2(F)_1$ where (e, h, f) is the standard \mathfrak{sl}_2 -triple. A **graded representation of \mathfrak{sl}_2** is a representation of \mathfrak{sl}_2 on a graded vector space $V = V_0 \oplus V_1$ such that $\mathfrak{sl}_2(F)_i(V_j) \subset V_{i+j}$ where $i, j \in \mathbb{Z}/2\mathbb{Z}$.

The following lemma is standard.

Lemma 7.7.7.

- (i) Every irreducible graded representation of \mathfrak{sl}_2 is irreducible (as a usual representation of \mathfrak{sl}_2).
- (ii) Every irreducible representation V of \mathfrak{sl}_2 admits exactly two gradings. In one grading the highest weight vector lies in V_0 and in the other grading it lies in V_1 .

Notation 7.7.8. Denote by V_λ^w be the irreducible graded representation of \mathfrak{sl}_2 with highest weight λ and highest weight vector of parity $w \in \mathbb{Z}/2\mathbb{Z}$.

Lemma 7.7.9. ⁸ Consider $\mathrm{Hom}((V_{\lambda_1}^{w_1}, V_{\lambda_2}^{w_2})^e)_0$ - the even part of the space of e -equivariant linear maps $V_{\lambda_1}^{w_1} \rightarrow V_{\lambda_2}^{w_2}$. Let $r_i := \dim V_{\lambda_i}^{w_i} = \lambda_i + 1$ and let

$$m := \mathrm{Tr}(h|_{\mathrm{Hom}((V_{\lambda_1}^{w_1}, V_{\lambda_2}^{w_2})^e)_0}) + \mathrm{Tr}(h|_{\mathrm{Hom}((V_{\lambda_2}^{w_2}, V_{\lambda_1}^{w_1})^e)_0}) - r_1 r_2.$$

Then

$$m = \begin{cases} -\min(r_1, r_2), & \text{if } r_1 \neq r_2 & (\text{mod } 2); \\ -2\min(r_1, r_2), & \text{if } r_1 \equiv r_2 \equiv 0 & (\text{mod } 2) \text{ and } w_1 = w_2; \\ 0, & \text{if } r_1 \equiv r_2 \equiv 0 & (\text{mod } 2) \text{ and } w_1 \neq w_2; \\ |r_1 - r_2| - 1, & \text{if } r_1 \equiv r_2 \equiv 1 & (\text{mod } 2) \text{ and } w_1 = w_2; \\ -(r_1 + r_2 - 1), & \text{if } r_1 \equiv r_2 \equiv 1 & (\text{mod } 2) \text{ and } w_1 \neq w_2; \end{cases}$$

This lemma follows by a direct computation from the following straightforward lemma.

Lemma 7.7.10. One has

$$(1) \quad \mathrm{Tr}(h|_{((V_\lambda^w)^e)_0}) = \begin{cases} \lambda, & \text{if } w = 0 \\ 0, & \text{if } w = 1 \end{cases}$$

$$(2) \quad (V_\lambda^w)^* = V_\lambda^{w+\lambda}$$

$$(3) \quad V_{\lambda_1}^{w_1} \otimes V_{\lambda_2}^{w_2} = \bigoplus_{i=0}^{\min(\lambda_1, \lambda_2)} V_{\lambda_1 + \lambda_2 - 2i}^{w_1 + w_2 + i}.$$

Proof of the Key Lemma. Let $V_0 := V_1 := F^n$. Let $V := V_0 \oplus V_1$ be a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space. We consider $\mathfrak{gl}_{2n}(F)$ as the $\mathbb{Z}/2\mathbb{Z}$ -graded Lie algebra $\mathrm{End}(V)$. Note that $\mathfrak{gl}_n(F) \times \mathfrak{gl}_n(F)$ is the even part of $\mathrm{End}(V)$ with respect to this grading. Consider V as a graded representation of the \mathfrak{sl}_2 triple (x, d, x_-) . Decompose V into graded irreducible representations W_i . Let $r_i := \dim W_i$ and w_i be the parity of the highest weight vector of W_i . Note that if r_i is even then $\dim(W_i \cap V_0) = \dim(W_i \cap V_1)$. If r_i is odd then $\dim(W_i \cap V_0) = \dim(W_i \cap V_1) + (-1)^{w_i}$. Since $\dim V_0 = \dim V_1$, we get that the number of indices i such that r_i is odd and $w_i = 0$ is equal to the number of indices i such that r_i is odd and $w_i = 1$. We denote this number by l . Now

$$\mathrm{Tr}(\mathrm{ad}(d)|_{(\mathfrak{gl}_n(F) \times \mathfrak{gl}_n(F))_x}) - 2n^2 = \mathrm{Tr}(d|_{(\mathrm{Hom}(V, V)^x)_0}) - 2n^2 = \frac{1}{2} \sum_{i,j} m_{ij},$$

where

$$m_{ij} := \mathrm{Tr}(d|_{(\mathrm{Hom}(W_i, W_j)^x)_0}) + \mathrm{Tr}(d|_{(\mathrm{Hom}(W_j, W_i)^x)_0}) - r_i r_j.$$

The m_{ij} can be computed using Lemma 7.7.9.

⁷This Lemma is similar to [JR96, §§3.2, Lemma 3.1]. The proofs are also similar.

⁸This Lemma is similar to [JR96, Lemma 3.2] but computes a different quantity.

As we see from the lemma, if either r_i or r_j is even then m_{ij} is non-positive and m_{ii} is negative. Therefore, if all r_i are even then we are done. Otherwise $l > 0$ and we can assume that all r_i are odd. Reorder the spaces W_i so that $w_i = 0$ for $i \leq l$ and $w_i = 1$ for $i > l$. Now

$$\begin{aligned} \sum_{1 \leq i, j \leq 2l} m_{ij} &= \sum_{i \leq l, j \leq l} (|r_i - r_j| - 1) + \sum_{i > l, j > l} (|r_i - r_j| - 1) - \sum_{i \leq l, j > l} (r_i + r_j - 1) - \sum_{i > l, j \leq l} (r_i + r_j - 1) = \\ &= \sum_{i \leq l, j \leq l} |r_i - r_j| + \sum_{i > l, j > l} |r_i - r_j| - \sum_{i \leq l, j > l} (r_i + r_j) - \sum_{i > l, j \leq l} (r_i + r_j) < \\ &< \sum_{i \leq l, j \leq l} (r_i + r_j) + \sum_{i > l, j > l} (r_i + r_j) - \sum_{i \leq l, j > l} (r_i + r_j) - \sum_{i > l, j \leq l} (r_i + r_j) = 0. \end{aligned}$$

The Lemma follows. \square

8. APPLICATIONS TO GELFAND PAIRS

8.1. Preliminaries on Gelfand pairs and distributional criteria.

In this section we recall a technique due to Gelfand-Kazhdan which allows to deduce statements in representation theory from statements on invariant distributions. For more detailed description see [AGS08, §2].

Definition 8.1.1. *Let G be a reductive group. By an **admissible representation of G** we mean an admissible representation of $G(F)$ if F is non-Archimedean (see [BZ76]) and admissible smooth Fréchet representation of $G(F)$ if F is Archimedean.*

We now introduce three a-priori distinct notions of Gelfand pair.

Definition 8.1.2. *Let $H \subset G$ be a pair of reductive groups.*

- We say that (G, H) satisfy **GP1** if for any irreducible admissible representation (π, E) of G we have

$$\dim \operatorname{Hom}_{H(F)}(E, \mathbb{C}) \leq 1.$$

- We say that (G, H) satisfy **GP2** if for any irreducible admissible representation (π, E) of G we have

$$\dim \operatorname{Hom}_{H(F)}(E, \mathbb{C}) \cdot \dim \operatorname{Hom}_H(\tilde{E}, \mathbb{C}) \leq 1.$$

- We say that (G, H) satisfy **GP3** if for any irreducible **unitary** representation (π, \mathcal{H}) of $G(F)$ on a Hilbert space \mathcal{H} we have

$$\dim \operatorname{Hom}_{H(F)}(\mathcal{H}^\infty, \mathbb{C}) \leq 1.$$

Property GP1 was established by Gelfand and Kazhdan in certain p -adic cases (see [GK75]). Property GP2 was introduced in [Gro91] in the p -adic setting. Property GP3 was studied extensively by various authors under the name **generalized Gelfand pair** both in the real and p -adic settings (see e.g. [vDP90], [vD86], [BvD94]).

We have the following straightforward proposition.

Proposition 8.1.3. $GP1 \Rightarrow GP2 \Rightarrow GP3$.

Remark 8.1.4. *It is not known whether some of these notions are equivalent.*

We will use the following theorem from [AGS08] which is a version of a classical theorem of Gelfand and Kazhdan (see [GK75]).

Theorem 8.1.5. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(\xi) = \xi$ for all bi $H(F)$ -invariant Schwartz distributions ξ on $G(F)$. Then (G, H) satisfies GP2.*

Corollary 8.1.6. *Any symmetric GK-pair satisfies GP2.*

In some cases, GP2 is known to be equivalent to GP1. For example, see Corollary 8.2.3 below.

8.2. Applications to Gelfand pairs.

Theorem 8.2.1. *Let G be a reductive group and let σ be an $\text{Ad}(G)$ -admissible anti-automorphism of G . Let θ be the automorphism of G defined by $\theta(g) := \sigma(g^{-1})$. Let (π, E) be an irreducible admissible representation of G .*

Then $\tilde{E} \cong E^\theta$, where \tilde{E} denotes the smooth contragredient representation and E^θ is E twisted by θ .

Proof. By Corollary 7.6.3, the characters of \tilde{E} and E^θ are identical. Since these representations are irreducible, this implies that they are isomorphic (see e.g. [Wal88, Theorem 8.1.5]). \square

Remark 8.2.2. *This theorem has an alternative proof using Harish-Chandra's Regularity Theorem, which says that the character of an admissible representation is a locally integrable function.*

Corollary 8.2.3. *Let $H \subset G$ be reductive groups and let τ be an $\text{Ad}(G)$ -admissible anti-automorphism of G such that $\tau(H) = H$. Then GP1 is equivalent to GP2 for the pair (G, H) .*

This corollary, together with Corollary 8.1.6 and Theorem 7.7.2 implies the following result.

Theorem 8.2.4. *The pair $(\text{GL}_{n+k}, \text{GL}_n \times \text{GL}_k)$ satisfies GP1.*

For non-Archimedean F this theorem is proven in [JR96].

Theorem 8.2.5. *Let E be a quadratic extension of F . Then the pair $(\text{GL}_n)_{E/F}, \text{GL}_n)$ satisfies GP1.*

For non-Archimedean F this theorem is proven in [Fli91].

Proof. By Theorem 7.6.2 this pair is tame. Hence it is enough to show that this symmetric pair is good. Consider the adjoint action of GL_n on itself. Let $x \in \text{GL}_n(E)^\sigma$ be semisimple. The stabilizer $(\text{GL}_n)_x$ is a product of groups of the form $(\text{GL}_n)_{F'/F}$ for some extensions F'/F . Hence $H^1(F, (\text{GL}_n)_x) = 0$. Therefore, by Corollary 7.1.5, the symmetric pair in question is good. \square

Part 3. Appendices

APPENDIX A. ALGEBRAIC GEOMETRY OVER LOCAL FIELDS

A.1. Implicit Function Theorems.

Definition A.1.1. *An analytic map $\phi : M \rightarrow N$ is called **étale** if $d_x\phi : T_xM \rightarrow T_{\phi(x)}N$ is an isomorphism for any $x \in M$. An analytic map $\phi : M \rightarrow N$ is called a **submersion** if $d_x\phi : T_xM \rightarrow T_{\phi(x)}N$ is onto for any $x \in M$.*

We will use the following version of the Inverse Function Theorem.

Theorem A.1.2 (cf. [Ser64], Theorem 2 in §9 of Chapter III in part II). *Let $\phi : M \rightarrow N$ be an étale map of analytic manifolds. Then it is locally an isomorphism.*

Corollary A.1.3. *Let $\phi : X \rightarrow Y$ be a morphism of (not necessarily smooth) algebraic varieties. Suppose that ϕ is étale at $x \in X(F)$. Then there exists an open neighborhood $U \subset X(F)$ of x such that $\phi|_U$ is a homeomorphism to its open image in $Y(F)$.*

For the proof see e.g. [Mum99, Chapter III, §5, proof of Corollary 2]. There, the proof is given for the case $F = \mathbb{C}$ but it works in general.

Remark A.1.4. *If F is Archimedean then one can choose U to be semi-algebraic.*

The following proposition is well known (see e.g. §10 of Chapter III in part II of [Ser64]).

Proposition A.1.5. *Any submersion $\phi : M \rightarrow N$ is open.*

Corollary A.1.6. *Lemma 2.3.4 holds. Namely, for any algebraic group G and a closed algebraic subgroup $H \subset G$ the subset $G(F)/H(F)$ is open and closed in $(G/H)(F)$.*

Proof. Consider the map $\phi : G(F) \rightarrow (G/H)(F)$ defined by $\phi(g) = gH$. Clearly, it is a submersion and its image is exactly $G(F)/H(F)$. Hence, $G(F)/H(F)$ is open. Since each $G(F)$ -orbit in $(G/H)(F)$ is open for the same reason, $G(F)/H(F)$ is also closed. \square

A.2. The Luna Slice Theorem.

In this subsection we formulate the Luna Slice Theorem and show how it implies Theorem 2.3.17. For a survey on the Luna Slice Theorem we refer the reader to [Dre00] and the original paper [Lun73].

Definition A.2.1 (cf. [Dre00]). *Let a reductive group G act on affine varieties X and Y . A G -equivariant algebraic map $\phi : X \rightarrow Y$ is called **strongly étale** if*

(i) $\phi/G : X/G \rightarrow Y/G$ is étale

(ii) ϕ and the quotient morphism $\pi_X : X \rightarrow X/G$ induce a G -isomorphism $X \cong Y \times_{Y/G} X/G$.

Definition A.2.2. *Let G be a reductive group and H be a closed reductive subgroup. Suppose that H acts on an affine variety X . Then $G \times_H X$ denotes $(G \times X)/H$ with respect to the action $h(g, x) = (gh^{-1}, hx)$.*

Theorem A.2.3 (Luna Slice Theorem). *Let a reductive group G act on a smooth affine variety X . Let $x \in X$ be G -semisimple.*

Then there exists a locally closed smooth affine G_x -invariant subvariety $Z \ni x$ of X and a strongly étale algebraic map of G_x spaces $\nu : Z \rightarrow N_{G_x, x}^X$ such that the G -morphism $\phi : G \times_{G_x} Z \rightarrow X$ induced by the action of G on X is strongly étale .

Proof. It follows from [Dre00, Proposition 4.18, Lemma 5.1 and Theorems 5.2 and 5.3], noting that one can choose Z and ν (in our notation) to be defined over F . \square

Corollary A.2.4. *Theorem 2.3.17 holds. Namely:*

Let a reductive group G act on a smooth affine variety X . Let $x \in X(F)$ be G -semisimple.

Then there exist

(i) *an open $G(F)$ -invariant B -analytic neighborhood U of $G(F)x$ in $X(F)$ with a G -equivariant B -analytic retract $p : U \rightarrow G(F)x$ and*

(ii) *a G_x -equivariant B -analytic embedding $\psi : p^{-1}(x) \hookrightarrow N_{G_x, x}^X(F)$ with an open saturated image such that $\psi(x) = 0$.*

Proof. Let Z , ϕ and ν be as in the last theorem.

Let $Z' := Z/G_x \cong (G \times_{G_x} Z)/G$ and $X' := X/G$. Consider the natural map $\phi' : Z'(F) \rightarrow X'(F)$. By Corollary A.1.3 there exists a neighborhood $S' \subset Z'(F)$ of $\pi_Z(x)$ such that $\phi'|_{S'}$ is a homeomorphism to its open image.

Consider the natural map $\nu' : Z'(F) \rightarrow N_{G_x, x}^X/G_x(F)$. Let $S'' \subset Z(F)$ be a neighborhood of $\pi_Z(x)$ such that $\nu'|_{S''}$ is an isomorphism to its open image. In case that F is Archimedean we choose S' and S'' to be semi-algebraic.

Let $S := \pi_Z^{-1}(S'' \cap S') \cap Z(F)$. Clearly, S is B -analytic.

Let $\rho : (G \times_{G_x} Z)(F) \rightarrow Z'(F)$ be the natural projection. Let $O = \rho^{-1}(S'' \cap S')$. Let $q : O \rightarrow (G/G_x)(F)$ be the natural map. Let $O' := q^{-1}(G(F)/G_x(F))$ and $q' := q|_{O'}$.

Now put $U := \phi(O')$ and put $p : U \rightarrow G(F)x$ be the morphism that corresponds to q' . Note that $p^{-1}(x) \cong S$ and put $\psi : p^{-1}(x) \rightarrow N_{G_x, x}^X(F)$ to be the imbedding that corresponds to $\nu|_S$. \square

APPENDIX B. SCHWARTZ DISTRIBUTIONS ON NASH MANIFOLDS

B.1. Preliminaries and notation.

In this appendix we will prove some properties of K -equivariant Schwartz distributions on Nash manifolds. We work in the notation of [AG08a], where one can read about Nash manifolds and Schwartz distributions over them. More detailed references on Nash manifolds are [BCR98] and [Shi87].

Nash manifolds are equipped with the **restricted topology**, in which open sets are open semi-algebraic sets. This is not a topology in the usual sense of the word as infinite unions of open sets are not necessarily open sets in the restricted topology. However, finite unions of open sets are open and therefore in the restricted topology we consider only finite covers. In particular, if $E \rightarrow M$ is a Nash vector bundle it means that there exists a finite open cover U_i of M such that $E|_{U_i}$ is trivial.

Notation B.1.1. *Let M be a Nash manifold. We denote by D_M the Nash bundle of densities on M . It is the natural bundle whose smooth sections are smooth measures. For the precise definition see e.g. [AG08a].*

An important property of Nash manifolds is

Theorem B.1.2 (Local triviality of Nash manifolds; [Shi87], Theorem I.5.12). *Any Nash manifold can be covered by a finite number of open submanifolds Nash diffeomorphic to \mathbb{R}^n .*

Definition B.1.3. *Let M be a Nash manifold. We denote by $\mathcal{G}(M) := \mathcal{S}^*(M, D_M)$ the **space of Schwartz generalized functions** on M . Similarly, for a Nash bundle $E \rightarrow M$ we denote by $\mathcal{G}(M, E) := \mathcal{S}^*(M, E^* \otimes D_M)$ the **space of Schwartz generalized sections** of E .*

*In the same way, for any smooth manifold M we denote by $C^{-\infty}(M) := \mathcal{D}(M, D_M)$ the **space of generalized functions** on M and for a smooth bundle $E \rightarrow M$ we denote by $C^{-\infty}(M, E) := \mathcal{D}(M, E^* \otimes D_M)$ the **space of generalized sections** of E .*

Usual L^1 -functions can be interpreted as Schwartz generalized functions but not as Schwartz distributions. We will need several properties of Schwartz functions from [AG08a].

Property B.1.4 ([AG08a], Theorem 4.1.3). $\mathcal{S}(\mathbb{R}^n) =$ *Classical Schwartz functions on \mathbb{R}^n .*

Property B.1.5 ([AG08a], Theorem 5.4.3). *Let $U \subset M$ be a (semi-algebraic) open subset, then*

$$\mathcal{S}(U, E) \cong \{\phi \in \mathcal{S}(M, E) \mid \phi \text{ is 0 on } M \setminus U \text{ with all derivatives}\}.$$

Property B.1.6 (see [AG08a], §5). *Let M be a Nash manifold. Let $M = \bigcup U_i$ be a finite open cover of M . Then a function f on M is a Schwartz function if and only if it can be written as $f = \sum_{i=1}^n f_i$ where $f_i \in \mathcal{S}(U_i)$ (extended by zero to M).*

Moreover, there exists a smooth partition of unity $1 = \sum_{i=1}^n \lambda_i$ such that for any Schwartz function $f \in \mathcal{S}(M)$ the function $\lambda_i f$ is a Schwartz function on U_i (extended by zero to M).

Property B.1.7 (see [AG08a], §5). *Let M be a Nash manifold and E be a Nash bundle over it. Let $M = \bigcup U_i$ be a finite open cover of M . Let $\xi_i \in \mathcal{G}(U_i, E)$ such that $\xi_i|_{U_j} = \xi_j|_{U_i}$. Then there exists a unique $\xi \in \mathcal{G}(M, E)$ such that $\xi|_{U_i} = \xi_i$.*

We will also use the following notation.

Notation B.1.8. *Let M be a metric space and $x \in M$. We denote by $B(x, r)$ the open ball with center x and radius r .*

B.2. Submersion principle.

Theorem B.2.1 ([AG08b], Theorem 2.4.16). *Let M and N be Nash manifolds and $s : M \rightarrow N$ be a surjective submersive Nash map. Then locally it has a Nash section, i.e. there exists a finite open cover $N = \bigcup_{i=1}^k U_i$ such that s has a Nash section on each U_i .*

Corollary B.2.2. *An étale map $\phi : M \rightarrow N$ of Nash manifolds is locally an isomorphism. That means that there exists a finite cover $M = \bigcup U_i$ such that $\phi|_{U_i}$ is an isomorphism onto its open image.*

Theorem B.2.3. *Let $p : M \rightarrow N$ be a Nash submersion of Nash manifolds. Then there exist a finite open (semi-algebraic) cover $M = \bigcup U_i$ and isomorphisms $\phi_i : U_i \cong W_i$ and $\psi_i : p(U_i) \cong V_i$ where $W_i \subset \mathbb{R}^{d_i}$ and $V_i \subset \mathbb{R}^{k_i}$ are open (semi-algebraic) subsets, $k_i \leq d_i$ and $p|_{U_i}$ correspond to the standard projections.*

Proof. The problem is local, hence without loss of generality we can assume that $N = \mathbb{R}^k$, M is an equidimensional closed submanifold of \mathbb{R}^n of dimension d , $d \geq k$, and p is given by the standard projection $\mathbb{R}^n \rightarrow \mathbb{R}^k$.

Let Ω be the set of all coordinate subspaces of \mathbb{R}^n of dimension d which contain N . For any $V \in \Omega$ consider the projection $pr : M \rightarrow V$. Define $U_V = \{x \in M \mid d_x pr \text{ is an isomorphism}\}$. It is easy to see that $pr|_{U_V}$ is étale and $\{U_V\}_{V \in \Omega}$ gives a finite cover of M . The theorem now follows from the previous corollary (Corollary B.2.2). \square

Theorem B.2.4. *Let $\phi : M \rightarrow N$ be a Nash submersion of Nash manifolds. Let E be a Nash bundle over N . Then*

(i) *there exists a unique continuous linear map $\phi_* : \mathcal{S}(M, \phi^*(E) \otimes D_M) \rightarrow \mathcal{S}(N, E \otimes D_N)$ such that for any $f \in \mathcal{S}(N, E^*)$ and $\mu \in \mathcal{S}(M, \phi^*(E) \otimes D_M)$ we have*

$$\int_{x \in N} \langle f(x), \phi_* \mu(x) \rangle = \int_{x \in M} \langle \phi^* f(x), \mu(x) \rangle.$$

In particular, we mean that both integrals converge.

(ii) *If ϕ is surjective then ϕ_* is surjective.*

Proof.

(i)

Step 1. Proof for the case when $M = \mathbb{R}^n$, $N = \mathbb{R}^k$, $k \leq n$, ϕ is the standard projection and E is trivial.

Fix Haar measure on \mathbb{R} and identify $D_{\mathbb{R}^l}$ with the trivial bundle for any l . Define

$$\phi_*(f)(x) := \int_{y \in \mathbb{R}^{n-k}} f(x, y) dy.$$

Convergence of the integral and the fact that $\phi_*(f)$ is a Schwartz function follows from standard calculus.

Step 2. Proof for the case when $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$ are open (semi-algebraic) subsets, ϕ is the standard projection and E is trivial.

Follows from the previous step and Property B.1.5.

Step 3. Proof for the case when E is trivial.

Follows from the previous step, Theorem B.2.3 and partition of unity (Property B.1.6).

Step 4. Proof in the general case.

Follows from the previous step and partition of unity (Property B.1.6).

(ii) The proof is the same as in (i) except of Step 2. Let us prove (ii) in the case of Step 2. Again, fix Haar measure on \mathbb{R} and identify $D_{\mathbb{R}^l}$ with the trivial bundle for any l . By Theorem B.2.1 and partition of unity (Property B.1.6) we can assume that there exists a Nash section $\nu : N \rightarrow M$. We can write ν in the form $\nu(x) = (x, s(x))$.

For any $x \in N$ define $R(x) := \sup\{r \in \mathbb{R}_{\geq 0} \mid B(\nu(x), r) \subset M\}$. Clearly, R is continuous and positive. By Tarski - Seidenberg principle (see e.g. [AG08a, Theorem 2.2.3]) it is semi-algebraic. Hence (by [AG08a, Lemma A.2.1]) there exists a positive Nash function $r(x)$ such that $r(x) < R(x)$. Let $\rho \in \mathcal{S}(\mathbb{R}^{n-k})$ such that ρ is supported in the unit ball and its integral is 1. Now let $f \in \mathcal{S}(N)$. Let $g \in C^\infty(M)$ defined by $g(x, y) := f(x)\rho((y - s(x))/r(x))/r(x)$ where $x \in N$ and $y \in \mathbb{R}^{n-k}$. It is easy to see that $g \in \mathcal{S}(M)$ and $\phi_* g = f$. \square

Notation B.2.5. *Let $\phi : M \rightarrow N$ be a Nash submersion of Nash manifolds. Let E be a bundle on N . We denote by $\phi^* : \mathcal{G}(N, E) \rightarrow \mathcal{G}(M, \phi^*(E))$ the dual map to ϕ_* .*

Remark B.2.6. *Clearly, the map $\phi^* : \mathcal{G}(N, E) \rightarrow \mathcal{G}(M, \phi^*(E))$ extends to the map $\phi^* : C^{-\infty}(N, E) \rightarrow C^{-\infty}(M, \phi^*(E))$ described in [AGS08, Theorem A.0.4].*

Proposition B.2.7. *Let $\phi : M \rightarrow N$ be a surjective Nash submersion of Nash manifolds. Let E be a bundle on N . Let $\xi \in C^{-\infty}(N)$. Suppose that $\phi^*(\xi) \in \mathcal{G}(M)$. Then $\xi \in \mathcal{G}(N)$.*

Proof. It follows from Theorem B.2.4 and Banach Open Map Theorem (see [Rud73, Theorem 2.11]). \square

B.3. Frobenius reciprocity.

In this subsection we prove Frobenius reciprocity for Schwartz functions on Nash manifolds.

Proposition B.3.1. *Let M be a Nash manifold. Let K be a Nash group. Let $E \rightarrow M$ be a Nash bundle. Consider the standard projection $p : K \times M \rightarrow M$. Then the map $p^* : \mathcal{G}(M, E) \rightarrow \mathcal{G}(M \times K, p^*E)^K$ is an isomorphism.*

This proposition follows from in [AG08b, Proposition 4.0.11].

Corollary B.3.2. *Let a Nash group K act on a Nash manifold M . Let E be a K -equivariant Nash bundle over M . Let $N \subset M$ be a Nash submanifold such that the action map $K \times N \rightarrow M$ is submersive. Then there exists a canonical map*

$$\text{HC} : \mathcal{G}(M, E)^K \rightarrow \mathcal{G}(N, E|_N).$$

Theorem B.3.3. *Let a Nash group K act on a Nash manifold M . Let N be a K -transitive Nash manifold. Let $\phi : M \rightarrow N$ be a Nash K -equivariant map.*

Let $z \in N$ be a point and $M_z := \phi^{-1}(z)$ be its fiber. Let K_z be the stabilizer of z in K . Let E be a K -equivariant Nash vector bundle over M .

Then there exists a canonical isomorphism

$$\text{Fr} : \mathcal{G}(M_z, E|_{M_z})^{K_z} \cong \mathcal{G}(M, E)^K.$$

Proof. Consider the map $a_z : K \rightarrow N$ given by $a_z(g) = gz$. It is a submersion. Hence by Theorem B.2.1 there exists a finite open cover $N = \bigcup_{i=1}^k U_i$ such that a_z has a Nash section s_i on each U_i . This gives an isomorphism $\phi^{-1}(U_i) \cong U_i \times M_z$ which defines a projection $p : \phi^{-1}(U_i) \rightarrow M_z$. Let $\xi \in \mathcal{G}(M_z, E|_{M_z})^{K_z}$. Denote $\xi_i := p^*\xi$. Clearly it does not depend on the section s_i . Hence $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ and hence by Property B.1.7 there exists $\eta \in \mathcal{G}(M, E)$ such that $\eta|_{U_i} = \xi_i$. Clearly η does not depend on the choices. Hence we can define $\text{Fr}(\xi) = \eta$.

It is easy to see that the map $\text{HC} : \mathcal{G}(M, E)^K \rightarrow \mathcal{G}(M_z, E|_{M_z})$ described in the last corollary gives the inverse map. \square

Since our construction coincides with the construction of Frobenius reciprocity for smooth manifolds (see e.g. [AGS08, Theorem A.0.3]) we obtain the following corollary.

Corollary B.3.4. *Part (ii) of Theorem 2.5.7 holds.*

B.4. K -invariant distributions compactly supported modulo K .

In this subsection we prove Theorem 4.0.5. Let us first remind its formulation.

Theorem B.4.1. *Let a Nash group K act on a Nash manifold M . Let E be a K -equivariant Nash bundle over M . Let $\xi \in \mathcal{D}(M, E)^K$ such that $\text{Supp}(\xi)$ is Nashly compact modulo K . Then $\xi \in \mathcal{S}^*(M, E)^K$.*

For the proof we will need the following lemmas.

Lemma B.4.2. *Let M be a Nash manifold. Let $C \subset M$ be a compact subset. Then there exists a relatively compact open (semi-algebraic) subset $U \subset M$ that includes C .*

Proof. For any point $x \in C$ choose an affine chart, and let U_x be an open ball with center at x inside this chart. Those U_x give an open cover of C . Choose a finite subcover $\{U_i\}_{i=1}^n$ and let $U := \bigcup_{i=1}^n U_i$. \square

Lemma B.4.3. *Let M be a Nash manifold. Let E be a Nash bundle over M . Let $U \subset M$ be a relatively compact open (semi-algebraic) subset. Let $\xi \in \mathcal{D}(M, E)$. Then $\xi|_U \in \mathcal{S}^*(U, E|_U)$.*

Proof. It follows from the fact that extension by zero $\text{ext} : \mathcal{S}(U, E|_U) \rightarrow C_c^\infty(M, E)$ is a continuous map. \square

Proof of Theorem B.4.1. Let $Z \subset M$ be a semi-algebraic closed subset and $C \subset M$ be a compact subset such that $\text{Supp}(\xi) \subset Z \subset KC$.

Let $U \supset C$ be as in Lemma B.4.2. Let $\xi' := \xi|_{KU}$. Since $\xi|_{M-Z} = 0$, it is enough to show that ξ' is Schwartz.

Consider the surjective submersion $m_U : K \times U \rightarrow KU$. Let

$$\xi'' := m_U^*(\xi') \in \mathcal{D}(K \times U, m_U^*(E))^K.$$

By Proposition B.2.7, it is enough to show that

$$\xi'' \in \mathcal{S}^*(K \times U, m_U^*(E)).$$

By Frobenius reciprocity, ξ'' corresponds to $\eta \in \mathcal{D}(U, E)$. It is enough to prove that $\eta \in \mathcal{S}^*(U, E)$. Consider the submersion $m : K \times M \rightarrow M$ and let

$$\xi''' := m^*(\xi) \in \mathcal{D}(K \times M, m^*(E)).$$

By Frobenius reciprocity, ξ''' corresponds to $\eta' \in \mathcal{D}(M, E)$. Clearly $\eta = \eta'|_U$. Hence by Lemma B.4.3, $\eta \in \mathcal{S}^*(U, E)$. \square

APPENDIX C. PROOF OF THE ARCHIMEDEAN HOMOGENEITY THEOREM

The goal of this appendix is to prove Theorem 5.1.7 for Archimedean F . First we remind its formulation.

Theorem C.0.1 (Archimedean Homogeneity). *Let V be a vector space over F . Let B be a non-degenerate symmetric bilinear form on V . Let M be a Nash manifold. Let $L \subset \mathcal{S}_{V \times M}^*(Z(B) \times M)$ be a non-zero subspace such that for all $\xi \in L$ we have $\mathcal{F}_B(\xi) \in L$ and $B\xi \in L$ (here B is interpreted as a quadratic form).*

Then there exists a non-zero distribution $\xi \in L$ which is adapted to B .

Till the end of the section we assume that F is Archimedean and we fix V and B .

First we will need some facts about the Weil representation. For a survey on the Weil representation in the Archimedean case we refer the reader to [RS78, §1].

(1) There exists a unique (infinitesimal) action π of $\mathfrak{sl}_2(F)$ on $\mathcal{S}^*(V)$ such that

$$(i) \pi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\xi = -i\pi Re(B)\xi \text{ and } \pi\left(\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right)\xi = -\mathcal{F}_B^{-1}(i\pi Re(B)\mathcal{F}_B(\xi)).$$

$$(ii) \text{ If } F = \mathbb{C} \text{ then } \pi\left(\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}\right) = \pi\left(\begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}\right) = 0$$

(2) It can be lifted to an action of the metaplectic group $\text{Mp}(2, F)$.

We will denote this action by Π .

(3) In case $F = \mathbb{C}$ we have $\text{Mp}(2, F) = \text{SL}_2(F)$ and in case $F = \mathbb{R}$ the group $\text{Mp}(2, F)$ is a connected 2-fold covering of $\text{SL}_2(F)$. We will denote by $\varepsilon \in \text{Mp}(2, F)$ the central element of order 2 satisfying $\text{SL}_2(F) = \text{Mp}(2, F)/\{1, \varepsilon\}$.

(4) In case $F = \mathbb{R}$ we have $\Pi(\varepsilon) = (-1)^{\dim V}$ and therefore if $\dim V$ is even then Π factors through $\text{SL}_2(F)$ and if $\dim V$ is odd then no nontrivial subrepresentation of Π factors through $\text{SL}_2(F)$. In particular if $\dim V$ is odd then Π has no nontrivial finite-dimensional representations, since every finite-dimensional representation of $\text{Mp}(2, F)$ factors through $\text{SL}_2(F)$.

(5) In case $F = \mathbb{C}$ or in case $\dim V$ is even we have $\Pi\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)\xi = \delta^{-1}(t)|t|^{-\dim V/2}\rho(t)\xi$ and

$$\Pi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\xi = \gamma(B)^{-1}\mathcal{F}_B\xi.$$

We also need the following straightforward lemma.

Lemma C.0.2. *Let (Λ, L) be a continuous finite-dimensional representation of $\text{SL}_2(\mathbb{R})$. Then there exists a non-zero $\xi \in L$ such that either*

$$\Lambda\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)\xi = \xi \text{ and } \Lambda\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\xi \text{ is proportional to } \xi$$

or

$$\Lambda\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)\xi = t\xi,$$

for all t .

Now we are ready to prove the theorem.

Proof of Theorem 5.1.7. Without loss of generality assume $M = pt$.

Let $\xi \in L$ be a non-zero distribution. Let $L' := U_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{R}))\xi \subset L$. Here, $U_{\mathbb{C}}$ means the complexified universal enveloping algebra.

We are given that $\xi, \mathcal{F}_B(\xi) \in \mathcal{S}_V^*(Z(B))$. By Lemma C.0.3 below this implies that $L' \subset \mathcal{S}^*(V)$ is finite-dimensional. Clearly, L' is also a subrepresentation of Π . Therefore by Fact (4), $F = \mathbb{C}$ or $\dim V$ is even. Hence Π factors through $\mathrm{SL}_2(F)$.

Now by Lemma C.0.2 there exists $\xi' \in L'$ which is B -adapted. \square

Lemma C.0.3. *Let V be a representation of \mathfrak{sl}_2 . Let $v \in V$ be a vector such that $e^k v = f^n v = 0$ for some n, k . Then the representation generated by v is finite-dimensional.*⁹

This lemma is probably well-known. Since we have not found any reference we include the proof.

Proof. The proof is by induction on k .

Base $k=1$:

It is easy to see that

$$e^l f^l v = l! \left(\prod_{i=0}^{l-1} (h-i) \right) v$$

for all l . This can be checked by direct computation, and also follows from the fact that $e^l f^l$ is of weight 0, hence it acts on the singular vector v by its Harish-Chandra projection which is

$$\mathrm{HC}(e^l f^l) = l! \prod_{i=0}^{l-1} (h-i).$$

Therefore $(\prod_{i=0}^{n-1} (h-i))v = 0$.

Hence $W := U_{\mathbb{C}}(h)v$ is finite-dimensional and h acts on it semi-simply. Here, $U_{\mathbb{C}}(h)$ denotes the universal enveloping algebra of h . Let $\{v_i\}_{i=1}^m$ be an eigenbasis of h in W . It is enough to show that $U_{\mathbb{C}}(\mathfrak{sl}_2)v_i$ is finite-dimensional for any i . Note that $e|_W = f^n|_W = 0$. Now, $U_{\mathbb{C}}(\mathfrak{sl}_2)v_i$ is finite-dimensional by the Poincare-Birkhoff-Witt Theorem.

Induction step:

Let $w := e^{k-1}v$. Let us show that $f^{n+k-1}w = 0$. Consider the element $f^{n+k-1}e^{k-1} \in U_{\mathbb{C}}(\mathfrak{sl}_2)$. It is of weight $-2n$, hence by the Poincare-Birkhoff-Witt Theorem it can be rewritten as a combination of elements of the form $e^a h^b f^c$ such that $c-a = n$ and hence $c \geq n$. Therefore $f^{n+k-1}e^{k-1}v = 0$.

Now let $V_1 := U_{\mathbb{C}}(\mathfrak{sl}_2)v$ and $V_2 := U_{\mathbb{C}}(\mathfrak{sl}_2)w$. By the base of the induction V_2 is finite-dimensional, by the induction hypotheses V_1/V_2 is finite-dimensional, hence V_1 is finite-dimensional. \square

APPENDIX D. LOCALIZATION PRINCIPLE

by Avraham Aizenbud, Dmitry Gourevitch and Eitan Sayag

In this appendix we formulate and prove the Localization Principle in the case of a reductive group G acting on a smooth affine variety X . This is of interest only for Archimedean F since for l -spaces, a more general version of this principle has been proven in [Ber84]. In [AGS09], we formulated without proof a Localization Principle in the setting of differential geometry. Admittedly, we currently do not have a proof of this principle in such a general setting. However, the current generality is sufficiently wide for all applications we encountered up to now, including the one in [AGS09].

Theorem D.0.1 (Localization Principle). *Let a reductive group G act on a smooth algebraic variety X . Let Y be an algebraic variety and $\phi : X \rightarrow Y$ be an affine algebraic G -invariant map. Let χ be a character of $G(F)$. Suppose that for any $y \in Y(F)$ we have $\mathcal{D}_{X(F)}((\phi^{-1}(y))(F))^{G(F), \chi} = 0$. Then $\mathcal{D}(X(F))^{G(F), \chi} = 0$.*

⁹For our purposes it is enough to prove this lemma for $k=1$.

Proof. Clearly, it is enough to prove the theorem for the case when X is affine, $Y = X/G$ and $\phi = \pi_X(F)$. By the Generalized Harish-Chandra Descent (Corollary 3.2.2), it is enough to prove that for any G -semisimple $x \in X(F)$, we have

$$\mathcal{D}_{N_{Gx,x}^X(F)}(\Gamma(N_{Gx,x}^X))^{G_x(F),\chi} = 0.$$

Let (U, p, ψ, S, N) be an analytic Luna slice at x . Clearly,

$$\mathcal{D}_{N_{Gx,x}^X(F)}(\Gamma(N_{Gx,x}^X))^{G_x(F),\chi} \cong \mathcal{D}_{\psi(S)}(\Gamma(N_{Gx,x}^X))^{G_x(F),\chi} \cong \mathcal{D}_S(\psi^{-1}(\Gamma(N_{Gx,x}^X)))^{G_x(F),\chi}.$$

By Frobenius reciprocity (Theorem 2.5.7),

$$\mathcal{D}_S(\psi^{-1}(\Gamma(N_{Gx,x}^X)))^{G_x(F),\chi} = \mathcal{D}_U(G(F)\psi^{-1}(\Gamma(N_{Gx,x}^X)))^{G(F),\chi}.$$

By Lemma 2.3.12,

$$G(F)\psi^{-1}(\Gamma(N_{Gx,x}^X)) = \{y \in X(F) \mid x \in \overline{G(F)y}\}.$$

Hence by Corollary 2.3.15, $G(F)\psi^{-1}(\Gamma(N_{Gx,x}^X))$ is closed in $X(F)$. Hence

$$\mathcal{D}_U(G(F)\psi^{-1}(\Gamma(N_{Gx,x}^X)))^{G(F),\chi} = \mathcal{D}_{X(F)}(G(F)\psi^{-1}(\Gamma(N_{Gx,x}^X)))^{G(F),\chi}.$$

Now,

$$G(F)\psi^{-1}(\Gamma(N_{Gx,x}^X)) \subset \pi_X(F)^{-1}(\pi_X(F)(x))$$

and we are given

$$\mathcal{D}_{X(F)}(\pi_X(F)^{-1}(\pi_X(F)(x)))^{G(F),\chi} = 0$$

for any G -semisimple x . □

Remark D.0.2. *An analogous statement holds for Schwartz distributions and the proof is the same.*

Corollary D.0.3. *Let a reductive group G act on a smooth algebraic variety X . Let Y be an algebraic variety and $\phi : X \rightarrow Y$ be an affine algebraic G -invariant submersion. Suppose that for any $y \in Y(F)$ we have $\mathcal{S}^*(\phi^{-1}(y))^{G(F),\chi} = 0$. Then $\mathcal{D}(X(F))^{G(F),\chi} = 0$.*

Proof. For any $y \in Y(F)$, denote $X(F)_y := (\phi^{-1}(y))(F)$. Since ϕ is a submersion, for any $y \in Y(F)$ the set $X(F)_y$ is a smooth manifold. Moreover, $d\phi$ defines an isomorphism between $N_{X(F)_y,z}^{X(F)}$ and $T_{Y(F),y}$ for any $z \in X(F)_y$. Hence the bundle $CN_{X(F)_y}^{X(F)}$ is a trivial $G(F)$ -equivariant bundle.

We know that

$$\mathcal{S}^*(X(F)_y)^{G(F),\chi} = 0.$$

Therefore for any k , we have

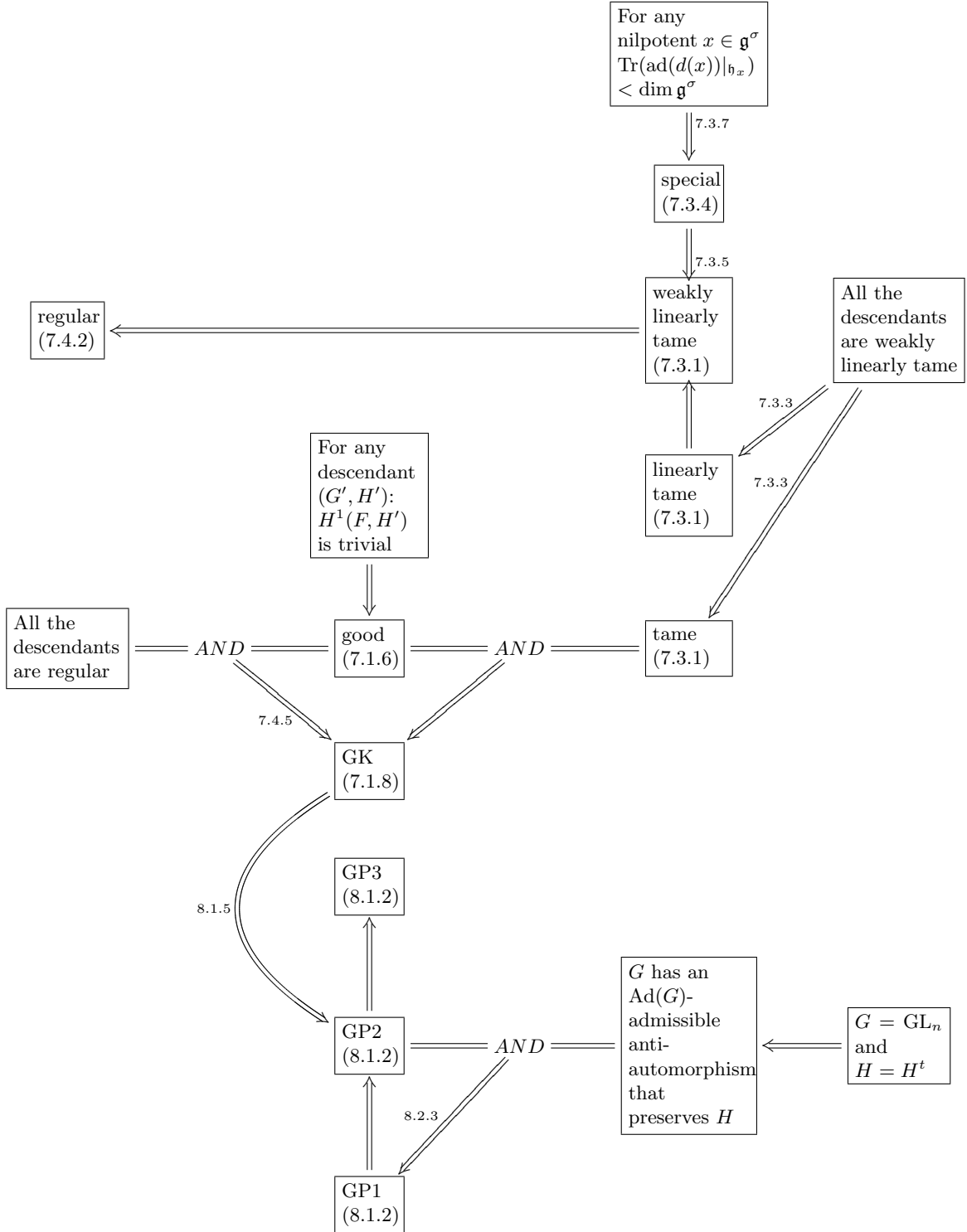
$$\mathcal{S}^*(X(F)_y, \text{Sym}^k(CN_{X(F)_y}^{X(F)}))^{G(F),\chi} = 0.$$

Thus by Theorem 2.5.6, $\mathcal{S}_{X(F)}^*(X(F)_y)^{G(F),\chi} = 0$. Now, by Theorem D.0.1 (and Remark D.0.2) this implies that $\mathcal{S}^*(X(F))^{G(F),\chi} = 0$. Finally, by Theorem 4.0.2 this implies $\mathcal{D}(X(F))^{G(F),\chi} = 0$. □

Remark D.0.4. *Theorem 4.0.1 and Corollary D.0.3 admit obvious generalizations to constant vector systems. The same proofs hold.*

APPENDIX E. DIAGRAM

The following diagram illustrates the interrelations of the various properties of a symmetric pair (G, H) . On the non-trivial implications we put the numbers of the statements that prove them. Near the important notions we put the numbers of the definitions which define those notions.



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