A SHORT PROOF OF HIRONAKA'S THEOREM ON FREENESS OF SOME HECKE MODULES

AVRAHAM AIZENBUD AND EITAN SAYAG

ABSTRACT. Let E/F be an unramified extension of non-archimedean local fields of residual characteristic different than 2.

We provide a simple geometric proof of a variation of a result of Hironaka ([Hir99]). Namely we prove that the module $\mathcal{S}(X)^{K_0}$ is free over the Hecke algebra $\mathcal{H}(SL_n(E), SL_n(O_E))$, where X is the space of unimodular Hermitian forms on E^n and O_E is the ring of integers in E.

CONTENTS

1. INTRODUCTION

Let F be a non-archimedean local field and let G be a reductive F-group. Suppose that X is an algebraic variety equipped with a G-action. Harmonic analysis on the $G(F)$-space $X(F)$, aims to study and decompose certain spaces of functions on $X(F)$ into simpler representations of $G(F)$.

A possible approach to this problem is to consider the structure of the $\mathcal{H}(G,K)-$ module $\mathcal{S}(X)^K$ of K-invariant compactly supported functions on X, where K is a compact open subgroup of $G(F)$ and $\mathcal{H}(G,K)$ is the Hecke algebra of $G(F)$ with respect to the subgroup K.

In the special case where $K = K_0$ is a maximal compact subgroup of G, the algebra $\mathcal{H}(G,K)$ is, by Satake's theorem, a finitely generated polynomial algebra. Thus, it is natural to study the structure of the module $\mathcal{S}(X)^{K_0}$ over this algebra using the language of commutative algebra. It turns out that in many cases, this module is free, a result with applications to multiplicities (see [Sa08]). Many special cases where studied ([Off], [Hir99], [MR09]) and general results are obtained in [Sa08] and [Sa13].

In this paper we prove the following result.

Theorem A. *Let E/F be an unramified quadratic extension of local non-archimedean fields of residual characteristic different than* 2. *Let $G = SL_n(E)$ and X be the space of Hermitian forms on E^n with determinant* 1. *Let K_0 be a maximal compact subgroup. Then $\mathcal{S}(X)^{K_0}$ is a free $\mathcal{H}(G,K_0)$ module of rank* $2^{\dim(V)-1}$.

Date: March 27, 2016.

Remark 1.0.1. *In [Hir99] a version of the above theorem concerning $GL(V)$ instead of $SL(V)$ was proven. It is not difficult to show that those two versions are equivalent.*

The proof in [Hir99] was spectral in that it was based on the explicit determination of the spherical functions on the space X associated to unramified representations. In our approach the proof is based solely on the geometry of the spherical space X and on the analysis of K_0 orbits.

1.1. **Idea of the proof.** The proof is based on a reduction technique we learned from [BL96] regarding filtered modules over filtered algebras. This technique allows to deduce the freeness of a module from the freeness of its associated graded. While classically one studies $\mathbb{Z}$-filtered modules, we need to adapt the technique to the case of $\mathbb{Z}^n$-filtered modules.

The filtrations we use on the spherical Hecke algebra and the spherical Hecke module $\mathcal{S}(X)^{K_0}$ are obtained from Cartan decompositions.

1.2. **Possible generalizations.** One can not expect that the conclusion of the Theorem holds for any spherical space. Nevertheless, we expect that for a large class of spherical spaces, one can find a subalgbera B of $\mathcal{H}(G, K_0)$ over which the module $\mathcal{S}(X(F))^{K_0}$ is free.

Our proof of Theorem A is based on certain geometric properties that we expect to holds for many symmetric spaces. Informally, we used the fact that the symmetric space X admits a nice Cartan decomposition. More precisely, we use a collection $\{g_\lambda \,|\lambda \in \Lambda^{++}\} \subset G$ and a collection $\{x_\lambda \,|\delta \in \Delta^{++}\} \subset X$, where $\Lambda^{++} \subset \Lambda$ is a Weyl chamber of the coweight lattice Λ and similarly for $\Delta^{++} \subset \Delta$ with the following properties:

- $G = \bigsqcup_{\lambda \in \Lambda^{++}} K_0 g_\lambda K_0$
- $X = \bigsqcup_{\delta \in \Delta^{++}} K_0 \cdot x_\delta$
- $K_0 g_\lambda K_0 \cdot K_0 g_\mu K_0 = \bigsqcup_{w \in W_\Lambda} K_0 g_{[w(\lambda)+\mu]} K_0$ where $\{[\gamma]\} := (W_\Lambda \cdot \gamma) \cap \Lambda^{++}$
- $K_0 g_\lambda K_0 \cdot K_0 x_\delta = \bigsqcup K_0 \cdot x_{[s(\lambda)+\mu]}$where $\{[\gamma]\} := (W_\Delta \cdot \gamma) \cap \Delta^{++}$ and $s : \Lambda \to \Delta$ is a certain symmetrization map.

We expect that under the above conditions, and certain technical conditions on the lattices Δ, Λ, it will be possible to adapt our argument to hold for any such X. In view of [Sa13] we expect those conditions to hold in many cases, but not for all symmetric pairs

1.3. **Acknowledgments.** : We would like to thank Omer Offen and Erez Lapid for conversations on [FLO2012] that motivated our interest in this problem. Part of the work on this paper was done during the research program *Multiplicities in representation theory* at the HIM.

2. Filtered modules and algebras

We first fix some terminology regarding filtered modules and algebras.

Definition 2.0.1.

- *For $i, j \in \mathbb{Z}^n$ we say that $j \le i$ if $i - j \in \mathbb{Z}^n_{\ge 0} := (\mathbb{Z}_{\ge 0})^n$.*
- *By a $\mathbb{Z}^n$-filtration on a vector space V we mean a collection of subspaces $F_i(V) \subset V$ for $i \in \mathbb{Z}^n$ s.t. there exist a $\mathbb{Z}^n$-grading $V = \bigoplus_{i \in \mathbb{Z}^n} F^0_i(V)$ with $F_i(V) = \bigoplus_{j \le i} F^0_j(V)$.*
- *For a $\mathbb{Z}^n$-filtrated vector space V, we denote $Gr^i_F(V) := F_i(V)/\sum_{j<i} F_j(V)$, and $Gr_F(V) := \bigoplus Gr^i_F(V)$.*
- *A $\mathbb{Z}^n$-filtration on an algebra A is a $\mathbb{Z}^n$-filtration $F^i(A)$ on the underlying vector space such that $F_i(A)F_j(A) \subset F_{i+j}(A)$. Note that in such a case $Gr_F(A)$ is $\mathbb{Z}^n$-graded algebra.*
- *Let $\phi : \mathbb{Z}^n \to \mathbb{Z}^m$ be a morphism. Let (A, F^0) be $\mathbb{Z}^n$-graded algebra. A ϕ-grading on an A-module M is a $\mathbb{Z}^m$-grading $G^0_i(M)$ on the underlying vector space M such that $F^0_i(A)G^0_j(M) \subset G^0_{\phi(i)+j}(M)$.*

- *Let $\phi : \mathbb{Z}^n \to \mathbb{Z}^m$ be a morphism and let (A, F) be a $\mathbb{Z}^n$ filtrated algebra. A ϕ-filtration on an A-module M is a $\mathbb{Z}^m$-filtration $G_i(M)$ on the underlying vector space such that $F_i(A)G_j(M) \subset G_{\phi(i)+j}(M)$. Note that in such a case $Gr_G(M)$ is a ϕ-graded module over $Gr_F(A)$.*

The following is an adaptation of a trick we learned from [BL96] (see Lemma 4.2).

Proposition 2.0.2. *Let $\phi : \mathbb{Z}^n \to \mathbb{Z}^m$ be a morphism.*

Let (M, G) be a ϕ-filtered module over a $\mathbb{Z}^n$-filtered commutative algebra (A, F). Assume that for any $i \notin \mathbb{Z}^n_{\geq 0}$ we have $Gr^i_F(A) = 0$ and for any $i \notin \mathbb{Z}^m_{\geq 0}$ we have $Gr^i_G(M) = 0$. Suppose that $Gr_G(M)$ is finitely generates free graded module over $Gr_F(A)$ (i.e. there exists finitely many homogenous elements that freely generate $Gr_G(M)$). Then M is a finitely generated free A-module.

More specifically if $\bar{m}_1, \dots, \bar{m}_k \in Gr_G(M)$ are homogenous elements that freely generates $Gr_G(M)$ over $Gr_F(A)$, then any lifts $m_1, \dots, m_k \in M$ freely generates M over A.

Proof.

Step 1. Proof in the case $m = n = 1$, $\phi = id$.
See, the proof of [BL96, Lemma 4.2].

Step 2. Proof in the case $\phi = id$.
The proof is by induction on n. Let $\bar{m}_1, \dots, \bar{m}_k \in Gr_G(M)$ be homogenous elements that freely generates $Gr_G(M)$ over $Gr_F(A)$ and $m_1, \dots, m_k \in M$ be there lifts.

For $i \in \mathbb{Z}$, we let $\bar{F}_i(A) = \sum_{k \in \mathbb{Z}^{(n-1)}} F_{(i,k)}(A)$. Similarly, we define $\bar{G}_i(M) = \sum_{k \in \mathbb{Z}^{(n-1)}} G_{(i,k)}(M)$. These are $\mathbb{Z}$-filtrations. Set $n_1, \dots, n_k \in Gr_{\bar{G}}(M)$ to be the reductions of $m_1, \dots, m_k \in M$.

By step 1 it is enough to show that $Gr_{\bar{G}}(M)$ is freely generated by $n_1, \dots, n_k$ over $Gr_{\bar{F}}(A)$. For this, define a $\mathbb{Z}^{(n-1)}$-filtrations on $Gr_{\bar{F}}(A)$ and $Gr_{\bar{G}}(M)$ by $\widetilde{F}_j(Gr^i_{\bar{F}}(A)) = F_{(i,j)}(A)/F_{(i,j)}(A) \cap \bar{F}_{i-1}(A)$ and $\widetilde{G}_j(Gr^i_{\bar{G}}(M)) = G_{(i,j)}(M)/G_{i,j}(M) \cap \bar{G}_{i-1}(M)$. The existence of the gradings $F^0_i(A), G^0_i(M)$ implies that $Gr_{\widetilde{F}}(Gr_{\bar{F}}(A)) \cong Gr_F(A)$ and $Gr_{\widetilde{G}}(Gr_{\bar{G}}(M)) \cong Gr_G(M)$. Furthermore, $\bar{m}_1, \dots, \bar{m}_k$ are the $\widetilde{G}$-reductions of $n_1, \dots, n_k$. Thus, the induction hypothesis implies that $Gr_{\bar{G}}(M)$ is freely generated by $n_1, \dots, n_k$ over $Gr_{\bar{F}}(A)$.

Step 3. The general case.
Define $\mathbb{Z}^m$-filtration on A by $\bar{F}_j(A) = \sum_{i \in \phi^{-1}(j)} F_j(A)$. By step 2, it is enough to show that $Gr_G(M)$ is freely generated by $\bar{m}_1, \dots, \bar{m}_k$ over $Gr_{\bar{F}}(A)$. For this we choose a gradation F^0_i s.t. $F_i(A) = \bigoplus_{j \leq i} F^0_j(A)$. This gives us a linear isomorphism $\psi : Gr_{\bar{F}}(A) \to Gr_F(A)$ s.t. $\psi(a)m = am$. We note that ψ is not necessary an algebra homomorphism. Since $Gr_G(M)$ is freely generated by $\bar{m}_1, \dots, \bar{m}_k$ over $Gr_F(A)$, this implies that $Gr_G(M)$ is freely generated by $\bar{m}_1, \dots, \bar{m}_k$ over $Gr_{\bar{F}}(A)$.

∎

3. Reduction to the Key Proposition

In this section we prove Theorem A. We will need some notations:

- Fix a natural number n. Let $H := H_n := SL_n$.
- Let E/F be an unramified quadratic extension of non-archimedean local fields of characteristic diffent than 2.
- We let $\tau : E \to E$ be the Galois involution.
- Let $G = G_n := Res^E_F(H_n)$ be the restriction of scalars of H to E (in particular $G(F) = H(E)$).
- We also fix $X := X_n$ the natural algebraic variety s.t. $X(F) = \{x \in G(E) | \tau(x^t) = x\}$.
- Let G act on X by

$$g \cdot x = gx\tau(g^t).$$

- Let $D \subset X$ be the subset of diagonal matrices.
- Finally, we let $T \subset G$ be the standard torus.

In the above notations, Theorem A reads as follows:

Theorem 3.0.1. *The module $\mathcal{S}(X(F))^{K_0}$ is free of rank 2^{n-1} over $\mathcal{H}(G, K_0)$ where $K_0 := SL(n, \mathcal{O}_E)$ is the standard maximal open subgroup of $G(F)$.*

Notation 3.0.2.

- *π a uniformizer in $\mathcal{O}_E$.*
- $q_F = |O_F/P_F|$, $q_E = |O_E/P_E|$.
- *Λ the weight lattice of G. We identify it with $\{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n | \lambda_1 + \cdots + \lambda_n = 0\}$.*
- $\Lambda^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda | \sum_{i=1}^k \lambda_i \geq 0 \ \forall k = 1, \dots, n\}$.
- $\Lambda^{++} = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda | \lambda_k - \lambda_{k-1} \leq 0 \ \forall k = 2, \dots, n\}$. *Note that $\Lambda^{++} \subset \Lambda^+$.*
- *for $\lambda \in \Lambda$ we set $\pi^\lambda := \lambda(\pi) \in G(F)$.*
- *for $\lambda \in \Lambda$ we set x_λ to be $\lambda(\pi)$ considered as an element in $X(F)$.*
- *Let $a_\lambda = e_{K_0 \delta_{\pi^\lambda} K_0} \in \mathcal{H}(G, K_0)$.*
- *Let $m_\lambda = e_{K_0 \delta_{x_\lambda}} \in \mathcal{S}(X(F))^{K_0}$.*
- *We denote $\lambda \geq \lambda'$ iff $\lambda - \lambda' \in \Lambda^+$. In this case, if $\lambda \neq \lambda'$ we denote $\lambda > \lambda'$.*

The following lemma is well known[1]

Lemma 3.0.3.

(1) The collection $\{\pi^\lambda | \lambda \in \Lambda^{++}\}$ is a complete set of representatives for the orbits of $K_0 \times K_0$ on G.
(2) The collection $\{x_\lambda | \lambda \in \Lambda^{++}\}$ is a complete set of representatives for the orbits of K_0 on X.

Corollary 3.0.4.

(1) The collection $\{a_\lambda | \lambda \in \Lambda^{++}\}$ is a basis for $\mathcal{H}(G, K_0)$.
(2) The collection $\{m_\lambda | \lambda \in \Lambda^{++}\}$ is a basis for $\mathcal{S}(X(F))^{K_0}$.

This Corollary leads naturally to the following filtration on the module $M := \mathcal{S}(X(F))^{K_0}$ and the Hecke algebra $A := \mathcal{H}(G, K_0)$.

Definition 3.0.5. *For $\lambda \in \Lambda$ we introduce the subspaces*

- $F_{\leq \lambda}(A) = Span_{\mathbb{C}}\{a_\mu | \mu \leq \lambda \ ; \mu \in \Lambda^{++}\}$, $\quad F_{<\lambda}(A) = Span_{\mathbb{C}}\{a_\mu | \mu < \lambda\}$
- $G_{\leq \lambda}(M) = Span_{\mathbb{C}}\{m_\mu | \mu \leq \lambda \ ; \mu \in \Lambda^{++}\}$, $\quad G_{<\lambda}(M) = Span_{\mathbb{C}}\{m_\mu | \mu < \lambda\}$

With this filtration we have the following Key Proposition:

Proposition 3.0.6.

(1) For every $\lambda \in \Lambda^{++}$ and $\mu \in \Lambda^{++}$ there exists a non-zero $p(\lambda, \mu) \in \mathbb{C}$ such that

$$a_\lambda a_\mu = p(\lambda, \mu) a_{\lambda+\mu} + r$$

with $r \in F_{<\lambda+\mu}(A)$.

(2) For every $\lambda \in \Lambda^{++}$ and $\mu \in \Lambda^{++}$ there exists a non-zero $q(\lambda, \mu) \in \mathbb{C}$ and we have

$$a_\lambda m_\mu = q(\lambda, \mu) m_{2\lambda+\mu} + \delta$$

where $\delta \in G_{<2\lambda+\mu}(M)$.

Part (1) is well known (see e.g. [Mac98, Chapter 5 (2.6)]). We postpone the proof of Part (2) to §4 and continue with the proof of Theorem 3.0.1

[1] Part (1) is the classical Cartan decomposition $G = K_0 A^{++} K_0$. A version of part (2) is proven in [Jac62].

Proof of Theorem 3.0.1. For $\lambda \in \mathbb{Z}^{n-1}$ denote $\tilde{F}_\lambda(A) = F_{\leq \tau(\lambda)}(A)$, $\tilde{G}_\lambda(M) = G_{\leq \tau(\lambda)}(M)$, where

$$\tau((\lambda_1, \dots, \lambda_{n-1})) = (\lambda_1, \lambda_2 - \lambda_1, \dots, \lambda_{n-1} - \lambda_{n-2}, -\lambda_{n-1}).$$

Let $\phi : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1}$ be given by $\phi(\lambda) = 2\lambda$. Proposition 3.0.6 implies that $\tilde{F}$ gives a structure of $\mathbb{Z}^n$-filtered algebra on A and ϕ-filtered module on M.

Applying Proposition 2.0.2 it is enough to show that $Gr_G(M)$ is finitely generated free $Gr_F(A)$-module. We now let $\bar{a}_\lambda, \bar{m}_\lambda$ be the reductions of a_λ, m_λ to the associated graded. By proposition 3.0.6 we get $\bar{a}_\lambda \bar{a}_\mu = p(\lambda, \mu)\bar{a}_{\lambda+\mu}$ and $\bar{a}_\lambda \bar{m}_\mu = q(\lambda, \mu)\bar{m}_{2\lambda+\mu}$. Let $L \subset \Lambda^{++}$ be a such that $\Lambda^{++} = \cup_{\ell \in L}(\ell + 2\Lambda^{++})$ is a disjoint covering. Clearly, the set $\{m_\ell | \ell \in L\}$ is a free basis of $Gr_G(M)$ over $Gr_F(A)$. This finishes the proof. ☐

4. Proof of Key Proposition 3.0.6

The proof of the proposition require an explicit version of Lemma 3.0.3. For this we require a definition.

Definition 4.0.7. *Let $V = E^n$ and $V_0 = F^n$*

(1) If L_1, L_2 are two O_E-lattices in V then we define

$$[L_1 : L_2] = \log_{q_E}(|L_1/(L_1 \cap L_2)||L_2/(L_1 \cap L_1)|^{-1})$$

(2) Let Q be a Hermitian form on V. Let $L \subset V_0$ be a lattice. Take an O_F basis $B = \{v_1, \dots, v_n\}$ to L. We define

$$\nu_L(Q) = \nu(det(Gram(B))) := \nu(det(Q(v_i, v_j))),$$

where ν is the valuation of E. This is independent of the choice of the basis.

Lemma 4.0.8. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^{++}$ and denote by $p_k = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ and let $q_k = \lambda_n + \lambda_{n-1} + \cdots + \lambda_{n-k+1}$.*

(1) Let $g \in K_0 \pi^\lambda K_0$. Then $p_k = \min_{W \in Grass(k,V)} [W \cap O_E^n : W \cap g O_E^n]$.

(2) Let $x \in K_0 x_\lambda$. Then $q_k = \min_{W \in Grass(k,V)} \nu_{O_E^n \cap W}(x|_W)$.

Proof. (1) We first note

$$\min_{W \in Grass(k,V)} [W \cap O_E^n : W \cap g O_E^n] = \min_{W \in Grass(k,V)} [W \cap O_E^n : W \cap \pi^\lambda O_E^n]$$

It remains to verify the statement of the lemma for $g = \pi^\lambda$. Clearly,

$$p_k \geq \min_{W \in Grass(k,V)} [W \cap O_E^n : W \cap \pi^\lambda O_E^n]$$

Thus it is enough to show that for any $W \in Grass(k, V)$ we have

$$p_k \leq [W \cap O_E^n : W \cap \pi^\lambda O_E^n]$$

For this we let $e_1, .., e_k$ be an O_E basis for $W \cap O_E^n$. Let $A \in Mat_{n \times k}(O_E)$ be the matrix whose i-the column is e_i, $i = 1, .., k$.

Denote by $r(A)$ the matrix obtained from A by reducing its elements to O/π. Since $e_1, ..., e_k$ is a basis we have $rank(r(A)) \geq k$ and we can find a $k \times k$ minor which is invertible in O_E. Explicitly, we have $\mathcal{I} = (i_1, i_2, .., i_k)$ such that the minor $M_{\mathcal{I},[1,k]}(A) \in O^\times$.

Notice that

$$\begin{aligned}[W \cap O_E^n : W \cap \pi^\lambda O_E^n] &= [Span_{O_E}(e_1, .., e_k) : \pi^\lambda(\pi^{-\lambda} W \cap O_E^n)] = \\ &= [Span_{O_E}(\pi^{-\lambda} e_1, .., \pi^{-\lambda} e_k) : \pi^{-\lambda} W \cap O_E^n] = \\ &= [Span_{O_E}(\pi^{-\lambda} e_1, .., \pi^{-\lambda} e_k) : Span_E(\pi^{-\lambda} e_1, .., \pi^{-\lambda} e_k) \cap O_E^n]\end{aligned}$$

Let $f_1, ..., f_k$ be an O_E-basis for $Span_E(\pi^{-\lambda}e_1, .., \pi^{-\lambda}e_k) \cap O_E^n$. Let $B \in Mat_{n\times k}(O_E)$ be the corresponding matrix as before.

Let $C \in Mat_{k\times k}(E)$ be such that $B = \pi^{-\lambda}AC$. Passing to the sub-matrix $B_{\mathcal{I},[1,..,k]}$ we have $B_{\mathcal{I},[1,..,k]} = diag(\pi^{-\lambda_{i_1}}, ..., \pi^{-\lambda_{i_k}})A_{\mathcal{I},[1,..,k]}C$. Thus $M_{\mathcal{I},[1,k]}(B) = \pi^{-\sum_{j=1}^{k}\lambda_{i_j}} M_{\mathcal{I},[1,k]}(A)det(C)$. Thus

$$0 \leq \nu(M_{\mathcal{I},[1,k]}(B)) = -\sum_{j=1}^{k}\lambda_{i_j} + \nu(M_{\mathcal{I},[1,k]}(A)) + \nu(det(C)) = -\sum_{j=1}^{k}\lambda_{i_j} + \nu(det(C))$$

Finally,

$$[W \cap O_E^n : W \cap \pi^{\lambda}O_E^n] = [Span_{O_E}(\pi^{-\lambda}e_1, ..., \pi^{-\lambda}e_k) : Span_{O_E}(f_1, ..., f_k)] = \nu(det(C)) \geq$$

$$\geq \sum_{j=1}^{k}\lambda_{i_j} \geq p_k$$

(2) as before, the only non-trivial part is to show that

$$\nu_{O_E^n \cap W}(x_\lambda|_W) \geq q_k.$$

If $x_\lambda|_W$ is degenerate this is obvious. So we will assume it is not. By Lemma 3.0.3 we can find a $x_\lambda|_W$-orthonormal basis $(e_1, \dots, e_k)$ of $O_E^n \cap W$ and a $x_\lambda|_{W^\perp}$-orthonormal basis $(e_{k+1}, \dots, e_n)$ of $O_E^n \cap W^\perp$. Let $\mu_i = \tau(e_i^t)x_\lambda e_i$. By Lemma 3.0.3 the collection $(\mu_1, \dots, \mu_n)$ coincides (up to reordering) with $(\lambda_1, \dots, \lambda_n)$ thus

$$\nu_{O_E^n \cap W}(x_\lambda|_W) = \mu_1 + \dots + \mu_k \geq \lambda_n + \dots + \lambda_{n-k+1} = q_k$$

□

Proof of Proposition 3.0.6 (2). Since $x_{2\lambda+\mu} \in \pi^\lambda K_0 x_\mu$, it is enough to show that $\pi^\lambda K_0 x_\mu \subset \bigcup_{\nu \leq 2\lambda+\mu} K_0 x_\nu$. Let $x \in K_0 x_\mu$.

By Lemma 4.0.8(2) we have to show

$$\min_{W \in Grass(i,V)} \nu_{W \cap O^n}(\pi^\lambda \cdot x|_W) \leq \sum_{j=n-i+1}^{n}(\mu_j + 2\lambda_j).$$

By Lemma 4.0.8 we have,

$$\min_{W \in Grass(i,V)} \nu_{O^n \cap W}(\pi^\lambda \cdot x|_W) = \min_{W \in Grass(i,V)} \nu_{\pi^\lambda O^n \cap \pi^\lambda W}(x|_{\pi^\lambda W}) =$$

$$= \min_{W \in Grass(i,V)} \nu_{\pi^\lambda O^n \cap W}(x|_W) = \min_{W \in Grass(i,V)} (2[O^n \cap W : \pi^\lambda O^n \cap W] + \nu_{O^n \cap W}(x|_W)) \leq$$

$$\leq 2 \min_{W \in Grass(i,V)}([O^n \cap W : \pi^\lambda O^n \cap W]) + \sum_{j=n-i+1}^{n}\mu_j = \sum_{j=n-i+1}^{n}(2\lambda_j + \mu_j).$$

□

References

[BL96] J.N. Bernstein and V. Lunts, *A simple proof of Kostant's theorem that $U(\mathfrak{g})$ is free over its center*, Amer. Jour. Math. v.118, no.5 (1996), pp. 979-987

[BZ76] I. N. Bernšteĭn and A. V. Zelevinskiĭ, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 5–70. MR MR0425030 (54 #12988)

[Bo76] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Inventiones Math. 35 (1976), 233259.

[Bus01] Colin J. Bushnell, *Representations of Reductive p-Adic Groups: Localization of Hecke Algebras and Applications.* J. London Math. Soc. (2001) **63**: 364-386; doi:10.1017/S0024610700001885

[BvD94] E. P. H. Bosman and G. Van Dijk, *A new class of Gelfand pairs*, Geom. Dedicata 50 (1994), 261 – 282. MR 1286380

[BT1] F. Bruhat and J. Tits, Groupes reductifs sur un corps local, Inst. Hautes Etudes Sci. Publ. Math. (1972), 5251. (French)

[BT2] Groupes reductifs sur un corps local. II. Schemas en groupes. Existence d'une donnee radicielle valuee, Inst. Hautes Etudes Sci. Publ. Math. (1984), 197376. (French) Bruhat, Tits

[Cas] W. Casselman, *The unramified principal series of p-adic groups. I. The spherical function*, Compositio Math. 40 (1980), no. 3, 387406.

[Del] P. Delorme, *Constant term of smooth H_ψ-spherical functions on a reductive p-adic group.* Trans. Amer. Math. Soc. **362** (2010), 933-955. See also http://iml.univ-mrs.fr/editions/publi2009/files/delorme_fTAMS.pdf.

[HC78] Harish-Chandra: *Admissible distributions on p-adic groups,* Queen's paper in pure and applied Math. 48, 1978, 281-346.

[FLO2012] B. Feigon, E. Lapid, O. Offen *On representations distigiuished by unitary groups*,Publ. Math. Inst. Hautes Études Sci., 115, N. 1, (2012), 185-323

[HW93] A.G. Helminck and S. P. Wang *On rationality properties of involutions of reductive groups* , Advances in Mathematics, vol. 99 (1993), 26-97.

[Hir99] Y. Hironaka *Spherical functions and local densities on Hermitian forms*, J. Math. Soc. Jpn., 51 (1999), 553-581. MR 1691493 (2000c:11064).

[Jac98] H. Jacquet *A theorem of density for Kloosterman integrals.* Asian J. Math. 2 (1998), no. 4, 759-778.

[Jac62] R. Jacobowitz, *Hermitian forms over local fields*, Amer. J. Math. 84 pp. 441-465, (1962).

[Lag08] N. Lagier , Terme constant de fonctions sur un espace symétrique réductif p-adique, J. of Funct. An., 254 (2008) 1088-1145.

[Lus83] G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Analysis and topology on singular spaces, II, III (Luminy, 1981), 208-229, Astrisque, 101-102, Soc. Math. France, Paris, 1983.

[MR09] Z. Mao and S. Rallis *A Plancherel formula for $Sp_{2n}/Sp_n x Sp_n$ and its application*, Compos. Math. 145 (2009), no. 2, 501-527.

[Ma77] H. Matsumoto, *Analyse Harmonique dans les Syst'emes de Tits Bornologiques de Type Affine*, Springer Lecture Notes N. 590, Berlin 1977.

[Mac98] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, Second Edition, 1998.

[Off] O. Offen *Relative spherical functions on p-adic symmetric spaces (three cases)*, Pacific J. Math. 215 (2004), no. 1, 97-149.

[Sa08] Y. Sakellaridis, *On the unramified spectrum of spherical varieties over p-adic fields*, Compositio Mathematica 144 (2008), no. 4, 978-1016.

[Sa13] Y. Sakellaridis, *Spherical functions on spherical varieties*, Amer. J. Math., 135(5):1291-1381, 2013.

[SV] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties.* ArXiv:1203.0039.

[SZ] B. Sun and C.-B. Zhu, *Multiplicity one theorems: the archimedean case*, arXiv:0903.1413[math.RT].

[KT08] S.I. Kato, K. Takano, *Subrepresentation theorem for p-adic symmetric spaces*, Int. Math. Res. Not. IMRN 2008, no. 11, Art. ID rnn028, 40 pp.

Avraham Aizenbud, Faculty of Mathematics and Computer Science, The Weizmann Institute of Science, ISRAEL.

E-mail address: aizner@gmail.com

URL: http://www.wisdom.weizmann.ac.il/~aizenr/

Eitan Sayag, Department of Mathematics, Ben-Gurion University of the Negev, ISRAEL

E-mail address: sayage@math.bgu.ac.il

URL: http://www.math.bgu.ac.il/~sayage/