SMOOTH TRANSFER OF KLOOSTERMAN INTEGRALS (THE ARCHIMEDEAN CASE)

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Abstract. We establish the existence of a transfer, which is compatible with Kloosterman integrals, between Schwartz functions on $\mathrm{GL}_n(\mathbb{R})$ and Schwartz functions on the variety of non-degenerate Hermitian forms. Namely, we consider an integral of a Schwartz function on $\mathrm{GL}_n(\mathbb{R})$ along the orbits of the two sided action of the groups of upper and lower unipotent matrices twisted by a non-degenerate character. This gives a smooth function on the torus. We prove that the space of all functions obtained in such a way coincides with the space that is constructed analogously when $\mathrm{GL}_n(\mathbb{R})$ is replaced with the variety of non-degenerate hermitian forms. We also obtain similar results for $\mathfrak{gl}_n(\mathbb{R})$.

The non-Archimedean case is done in [Jac03a] and our proof is based on the ideas of this work. However we have to face additional difficulties that appear only in the Archimedean case.

Those results are crucial for the comparison of the Kuznetsov trace formula and the relative trace formula of GL_n with respect to the maximal unipotent subgroup and the unitary group, as done in [Jac05b] and [FLO].

Contents

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1. Introduction

Let N^n be the subgroup of upper triangular matrices in GL_n with unit diagonal, and let A^n be the group of invertible diagonal matrices. We define a character $\theta : N^n(\mathbb{R}) \to \mathbb{C}^\times$ by

$$\theta(u) = \exp(i \sum_{k=1}^{n-1} u_{k,k+1}).$$

Let $\mathcal{S}(GL_n(\mathbb{R}))$ be the space of Schwartz functions on $GL_n(\mathbb{R})$. We define a map $\Omega : \mathcal{S}(GL_n(\mathbb{R})) \to C^\infty(A^n)$ by

$$\Omega(\Phi)(a) := \int_{(u_1,u_2)\in N^n(\mathbb{R})\times N^n(\mathbb{R})} \Phi(u_1^t a u_2)\theta(u_1 u_2) du_1 du_2.$$

Similarly, we let $S^n(\mathbb{C})$ be the space of non-degenerate Hermitian matrices $n \times n$. We define a map $\Omega : \mathcal{S}(S^n(\mathbb{C})) \to C^\infty(A^n)$ by

$$\Omega(\Psi)(a) := \int_{u\in N^n(\mathbb{C})} \Psi(\overline{u}^t a u)\theta(u\overline{u}) du.$$

We say that $\Phi \in \mathcal{S}(GL_n(\mathbb{R}))$ *matches* $\Psi \in \mathcal{S}(S^n(\mathbb{C}))$ if for every $a \in A^n(F)$, we have

$$\Omega(\Phi)(a) = \gamma(a)\Omega(\Psi)(a),$$

where

$$\gamma(a) := sign(a_1)sign(a_1 a_2)...sign(a_1 a_2, ..., a_{n-1}) \text{ for } a = diag(a_1, a_2, ..., a_n).$$

The main theorem of this paper is

Theorem A. *For every $\Phi \in \mathcal{S}(GL_n(\mathbb{R}))$ there is a matching $\Psi \in \mathcal{S}(S^n(\mathbb{C}))$, and conversely.*

We also prove a similar theorem for $\mathfrak{gl}_n$.

We also consider non-regular orbital integrals and prove that if two functions match then their non-regular orbital integrals are also equal (up to a suitable transfer factor). This implies in particular that regular orbital integrals are dense in all orbital integrals.

1.1. Motivation and related works.

The motivation for this paper comes from the relative trace formula, and its use for comparison of representation theories of different groups. For a background on the relevant relative trace formula and its connection with the problems studied in this paper we refer the reader to [Jac05b, FLO].

Now we will give a brief description of this connection.

The study of $Ad(G)$-invariant distributions on a group G is closely related to the study of representation theory of G. One can replace the study of $Ad(G)$-invariant distributions on G by the study of $Ad(G)$-orbital integrals. Similarly, the study of two-sided $H_1 \times H_2$-orbital integrals on G is closely related to harmonic analysis on G/H_1 and on G/H_2. Here, $H_1, H_2 < G$ are subgroups. Those relations become efficient in view of the trace formula and the relative trace formula.

Therefore in order to compare the harmonic analysis on G/H_1 and G/H_2 to harmonic analysis on G'/H_1' and G'/H_2' using the relative trace formula one has to compare the orbital integrals. Here, $H_1', H_2' < G'$ is another triple of groups which is believed to be connected to $H_1, H_2 < G$.

The present work is relevant to the comparison of the triple $GL_n(F), N^n(F), N^n(F)$ and the triple $GL_n(E), N^n(E), U(E)$, where E is a quadratic extension of F and U denotes the unitary group. Since one has to use the adelic trace formula in order to efficiently relate orbital integrals to harmonic analysis, one has to use the corresponding fundamental lemma, conjectured in [JY90] and proven in [Ngo97, Ngo99, Jac04].

Also, in order to turn the local comparison into an adelic comparison one needs the local comparison over both Archimedean and non-Archimedean local fields. In this paper we consider only the Archimedean case. The non-Archimedean counterpart of this paper is done in [Jac03a, Jac03b] and our proof is based on the ideas of these works. However we have to face additional difficulties that appear only in the Archimedean case.

In the case of $\mathrm{GL}(2,\mathbb{R})$, Theorem A was proven in [Jac05a], using different methods.

1.2. **A sketch of the proof.**

First we show that the theorem for $\mathfrak{gl}_n$ implies the theorem for GL_n. Then we prove the theorem for $\mathfrak{gl}_n$ by induction. We construct certain open sets $O_i \subset \mathfrak{gl}_n(\mathbb{R})$ (for their definition see §§3.1) and use the *intermediate Kloosterman integrals* in order to describe $\Omega(\mathcal{S}(O_i))$ in terms of $\Omega(\mathcal{S}(GL_i(\mathbb{R})))$ and $\Omega(\mathcal{S}(gl_{n-i}(\mathbb{R})))$. This gives a smooth matching for $\mathcal{S}(O_i)$ by the induction hypothesis. We denote $U := \bigcup O_i$ and $Z := \mathfrak{gl}_n(\mathbb{R}) - U$ and obtain by partition of unity smooth matching for $\mathcal{S}(U)$.

Then we use an important fact. Namely, if Φ matches Ψ then the Fourier transform of Φ matches the Fourier transform of Ψ multiplied by a constant. This is proven in [Jac03a] in the non-Archimedean case and the same proof holds in the Archimedean case. The proof of this fact is based on an explicit formula for the Kloosterman integral of the Fourier transform of Φ in terms of the Kloosterman integral of Φ (see Theorem 3.2.6). We call the right hand side of this formula the Jacquet transform of $\Omega(\Phi)$.

In order to complete the proof of the main theorem we prove the following Key Lemma.

Lemma B. *Let $N^n \times N^n$ act on $\mathfrak{gl}_n$ by $x \mapsto u_1^t x u_2$. Let χ denote the character of $N^n \times N^n$ defined by $\chi(u_1, u_2) = \theta(u_1 u_2)$.*

Then any function in $\mathcal{S}(\mathfrak{gl}_n(\mathbb{R}))$ can be written as a sum $f + g + h$ s.t. f is a Schwartz function on U, the Fourier transform of g is a Schwartz function on U and h is a function that annihilates any $(N^n \times N^n, \chi)-$ equivariant distribution on $\mathfrak{gl}_n(\mathbb{R})$ and in particular $\Omega(h) = 0$.

Remark C. *Our proof of the main theorem gives a recursive description of the space $\Omega(\mathcal{S}(gl_n(\mathbb{R})))$. Namely we prove that this space is the sum of the space $\Omega(\mathcal{S}(U))$ and its Jacquet transform. The space $\Omega(\mathcal{S}(U))$ is the sum of the spaces $\Omega(O_i)$ and those spaces can be described in terms of $\Omega(\mathcal{S}(gl_i(\mathbb{R})))$ and $\Omega(\mathcal{S}(gl_{n-i}(\mathbb{R})))$.*

1.3. **The spaces of functions considered.**

Since the proof relies on Fourier transform, in the Archimedean case it would not be appropriate to consider the space of smooth compactly supported functions. Therefore we had to work with Schwartz functions. Theories of Schwartz functions were developed by various authors in various generalities. We chose for this problem the version developed in [AG08, AG10] in the generality of Nash (i.e. smooth semi-algebraic) manifolds. In Appendices A and B of the present paper we develop further the tools for working with Schwartz functions from [AG08, AG10] and [AG09, Appendix B], for the purposes of this paper.

1.4. **Difficulties that we encounter in the Archimedean case.**

Roughly speaking, most of the additional difficulties in the Archimedean case come from the fact that the space of Schwartz functions in the Archimedean case is a topological vector space unlike the space of Schwartz functions in the non-Archimedean case which is just a vector space. Part of those difficulties are technical and can be overcome using the theory of nuclear Fréchet spaces. However there are more essential difficulties in the Key Lemma. Namely, in the non-Archimedean case the Key lemma is equivalent to the following one

Lemma D. *Any $(N^n \times N^n, \chi)$-equivariant distribution on $\mathfrak{gl}_n(\mathbb{R})$ supported on Z, whose Fourier transform is also supported on Z, vanishes.*

Note that even this lemma is harder in the Archimedean since we have to deal with transversal derivatives. However, this difficulty is overcome using the fact that the transversal derivatives are controlled by the action of stabilizer of a point on the normal space to its orbit. This actio is rather simple since it is an algebraic action of a unipotent group.

The main difficulty, though, is that in the Archimedean case Lemma D in not equivalent to Lemma B but only to the following weak version of it

Lemma E. *Any function in $\mathcal{S}(\mathfrak{gl}_n(\mathbb{R}))$ can be approximated by a sum $f+g+h$ s.t. f is a Schwartz function on U, the Fourier transform of g is a Schwartz function on U and h is a function that annihilates any $(N^n \times N^n, \chi)-$ equivariant distribution on $\mathfrak{gl}_n(\mathbb{R})$ and in particular $\Omega(f)=0$.*

We believe that the reason that the Key Lemma holds is a part of a general phenomenon. To describe this phenomenon note that a statement concerning equivariant distributions can be reformulated to a statement concerning closure of subspaces of Schwartz functions. The phenomenon is that in many cases this statement holds without the need to consider the closure. We discuss two manifestations of this phenomenon in §§§2.2.2 and 2.2.3, and prove them in appendices B and A.2. The proofs there remind in their spirit the proof of the classical Borel Lemma.

1.5. **Contents of the paper.**

In §2 we fix notational conventions and list the basic facts on Schwartz functions and nuclear Fréchet spaces that we will use.

In §3 we prove the main result. In §§3.1 we introduce the notation that we will use to discuss our problem, and reformulate the main result in this notation. In §§3.2 we introduce the main ingredients of the proof: description of $\Omega(\mathcal{S}(O_i))$ using intermediate Kloosterman integrals, inversion formula that connects Fourier transform to Kloosterman integrals, and the Key lemma. In §§3.3 we deduce the main result, Theorem A, from the main ingredients.

In §4 we prove the inversion formula.

In §5 we prove the Key lemma.

In §6 we consider non-regular orbital integrals, define matching for them and prove that if two functions match then their non-regular orbital integrals also match.

In appendices A and B we give some complementary facts about Nash manifolds and Schwartz functions on them and prove an analog of Dixmier - Malliavin Theorem and prove dual versions of special cases of uncertainty principle and localization principle. Those versions are two manifestations of the phenomenon described above.

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2. Preliminaries

2.1. **General notation.**

- All the algebraic varieties and algebraic groups we consider in this paper are real.
- For a group G acting on a set X and a point $x \in X$ we denote by Gx or by $G(x)$ the orbit of x, by G_x the stabilizer of x and by X^G the set of G-fixed points in X.
- For Lie groups G or H we will usually denote their Lie algebras by $\mathfrak{g}$ and $\mathfrak{h}$.
- An action of a Lie algebra $\mathfrak{g}$ on a (smooth, algebraic, etc) manifold M is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Note that an action of a (Lie, algebraic, etc) group on M defines an action of its Lie algebra on M.

- For a Lie algebra $\mathfrak{g}$ acting on M, an element $\alpha \in \mathfrak{g}$ and a point $x \in M$ we denote by $\alpha(x) \in T_xM$ the value at point x of the vector field corresponding to α. We denote by $\mathfrak{g}x \subset T_xM$ or by $\mathfrak{g}(x)$ the image of the map $\alpha \mapsto \alpha(x)$ and by $\mathfrak{g}_x \subset \mathfrak{g}$ its kernel.
- For a Lie algebra (or an associative algebra) $\mathfrak{g}$ acting on a vector space V and a subspace $L \subset V$, we denote by $\mathfrak{g}L \subset V$ the image of the action map $\mathfrak{g} \otimes L \to V$.
- For a representation V of a Lie algebra $\mathfrak{g}$ we denote by $V^{\mathfrak{g}}$ the space of $\mathfrak{g}$-invariants and by $V_{\mathfrak{g}} := V/\mathfrak{g}V$ the space of $\mathfrak{g}$-coinvariants.
- For manifolds $L \subset M$ we denote by $N_L^M := (T_M|_L)/T_L$ the normal bundle to L in M.
- Denote by $CN_L^M := (N_L^M)^*$ the conormal bundle.
- For a point $y \in L$ we denote by $N_{L,y}^M$ the normal space to L in M at the point y and by $CN_{L,y}^M$ the conormal space.
- By bundle we always mean a vector bundle.
- For a manifold M we denote by $C^\infty(M)$ the space of infinitely differentiable functions on M, equipped with the standard topology.

2.2. **Schwartz functions on Nash manifolds.**

We will require a theory of Schwartz functions on Nash manifolds as developed e.g. in [AG08]. Nash manifolds are smooth semi-algebraic manifolds but in the present work, except of Appendix A, only smooth real algebraic manifolds are considered. Therefore the reader can safely replace the word *Nash* by *smooth real algebraic* in the body of the paper.

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG08]. We will use the following notations.

Notation 2.2.1. *Let X be a Nash manifold. Denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X.*

We will need several properties of Schwartz functions from [AG08].

Property 2.2.2 ([AG08], Theorem 4.1.3)**.** *$\mathcal{S}(\mathbb{R}^n)$ = Classical Schwartz functions on $\mathbb{R}^n$.*

Property 2.2.3 ([AG08], Theorem 5.4.3)**.** *Let $U \subset M$ be an open Nash submanifold, then*

$$\mathcal{S}(U) \cong \{\phi \in \mathcal{S}(M) | \quad \phi \text{ is } 0 \text{ on } M \setminus U \text{ with all derivatives}\}.$$

In this paper we will consider $\mathcal{S}(U)$ as a subspace of $\mathcal{S}(X)$.

Property 2.2.4 (see [AG08], §5)**.** *Let M be a Nash manifold. Let $M = \bigcup_{i=1}^n U_i$ be a finite cover of M by open Nash submanifolds. Then a function f on M is a Schwartz function if and only if it can be written as $f = \sum_{i=1}^n f_i$ where $f_i \in \mathcal{S}(U_i)$ (extended by zero to M).*

Moreover, there exists a smooth partition of unity $1 = \sum_{i=1}^n \lambda_i$ such that for any Schwartz function $f \in \mathcal{S}(M)$ the function $\lambda_i f$ is a Schwartz function on U_i (extended by zero to M).

Property 2.2.5 (see [AG08], §5)**.** *Let $Z \subset M$ be a Nash closed submanifold. Then restriction maps $\mathcal{S}(M)$ onto $\mathcal{S}(Z)$.*

Property 2.2.6 ([AG09], Theorem B.2.4)**.** *Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Let E be a Nash bundle over N. Fix Nash measures μ on M and ν on N.*

Then

(i) there exists a unique continuous linear map $\phi_* : \mathcal{S}(M) \to \mathcal{S}(N)$ such that for any $f \in \mathcal{S}(N)$ and $g \in \mathcal{S}(M)$ we have

$$\int_{x \in N} f(x)\phi_* g(x) d\nu = \int_{x \in M} (f(\phi(x)))g(x) d\mu.$$

In particular, we mean that both integrals converge.

(ii) If ϕ is surjective then ϕ_* is surjective.

In fact

$$\phi_* g(x) = \int_{z \in \phi^{-1}(x)} g(z) d\rho$$

for an appropriate measure ρ.

We will need the following analog of Dixmier - Malliavin theorem.

Property 2.2.7. *Let* $\phi : M \to N$ *be a Nash map of Nash manifolds. Then multiplication defines an onto map* $\mathcal{S}(M) \otimes \mathcal{S}(N) \twoheadrightarrow \mathcal{S}(M)$.

For proof see Theorem A.1.1.

We will also need the following notion.

Notation 2.2.8. *Let* $\phi : M \to N$ *be a Nash map of Nash manifolds. We call a function* $f \in C^\infty(M)$ *Schwartz along the fibers of* ϕ *if for any Schwartz function* $g \in \mathcal{S}(N)$, *we have* $(g \circ \phi) f \in \mathcal{S}(M)$.

We denote the space of such functions by $\mathcal{S}^{\phi,N}(M)$. *If there is no ambiguity we will sometimes denote it by* $\mathcal{S}^{\phi}(M)$ *or by* $\mathcal{S}^N(M)$. *We define the topology on* $\mathcal{S}^{\phi}(M)$ *using the following system of semi-norms: for any seminorms* α *on* $\mathcal{S}(N)$ *and* β *on* $\mathcal{S}(M)$ *we define*

$$\mathfrak{N}^{\alpha}_{\beta}(f) := \sup_{g \in \mathcal{S}(N) | \alpha(g) < 1} \beta(f(g \circ \phi)).$$

We will use the following corollary of Property 2.2.6.

Corollary 2.2.9. *Let* $\phi : M \to N$ *be a Nash map and* $\psi : L \to M$ *be a Nash submersion. Fix Nash measures on* L *and* M. *Then there is a natural continuous linear map* $\phi_* : \mathcal{S}^N(L) \to \mathcal{S}^N(M)$.

Remark 2.2.10. *Let* $\phi : M \to N$ *be a Nash map of Nash manifolds. Let* $V \subset N$ *be a dense open Nash submanifold. Let* $U := \phi^{-1}(V)$. *Suppose that* U *is dense in* M. *Then we have embeddings*

$$\mathcal{S}(M) \hookrightarrow \mathcal{S}^{\phi,N}(M) \hookrightarrow \mathcal{S}^{\phi,V}(U).$$

In this paper we will view $\mathcal{S}(M)$ *and* $\mathcal{S}^{\phi,N}(M)$ *as subspaces of* $\mathcal{S}^{\phi,V}(U)$.

2.2.1. *Fourier transform.*

Notation 2.2.11. *Let* V *be a finite dimensional real vector space. Let* B *be a non-degenerate bilinear form on* V *and* ψ *be a non-trivial additive character of* $\mathbb{R}$. *Then* B *and* ψ *define Fourier transform with respect to the self-dual Haar measure on* V. *We denote it by* $\mathcal{F}_{B,\psi} : \mathcal{S}(V) \to \mathcal{S}(V)$. *If there is no ambiguity, we will omit* B *and* ψ. *We will also denote by* $\mathcal{F}^*_{B,\psi} : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$ *the dual map.*

We will use the following trivial observation.

Lemma 2.2.12. *Let* V *be a finite dimensional real vector space. Let a Nash group* G *act linearly on* V. *Let* B *be a* G*-invariant non-degenerate symmetric bilinear form on* V. *Let* ψ *be a non-trivial additive character of* $\mathbb{R}$. *Then* $\mathcal{F}_{B,\psi}$ *commutes with the action of* G.

2.2.2. *Dual uncertainty principle.*

Theorem 2.2.13. *Let* V *be a finite dimensional real vector space. Let* B *be a non-degenerate bilinear form on* V *and* ψ *be a non-trivial additive character of* $\mathbb{R}$. *Let* $L \subset V$ *be a subspace. Suppose that* $L^\perp \subsetneqq L$. *Then*

$$\mathcal{S}(V - L) + \mathcal{F}(\mathcal{S}(V - L)) = \mathcal{S}(V).$$

For proof see Appendix A.2.

Remark 2.2.14. *It is much easier to prove that*

$$\overline{\mathcal{S}(V - L) + \mathcal{F}(\mathcal{S}(V - L))} = \mathcal{S}(V)$$

since this is equivalent to the fact that there are no distributions on V *supported in* L *with Fourier transform supported in* L.

2.2.3. *Coinvariants in Schwartz functions.*

Theorem 2.2.15. *Let a connected algebraic group G act on a real algebraic manifold X. Let Z be a G-invariant Zariski closed subset of X. Let $\mathfrak{g}$ be the Lie algebra of G. Let χ be a unitary character of G. Suppose that for any $\mathfrak{z} \in Z$ and $k \in \mathbb{Z}_{\geq 0}$ we have*

$$(\chi \otimes \mathrm{Sym}^k(CN_{Gz,z}^X) \otimes ((\Delta_G)|_{G_z}/\Delta_{G_z}))_{\mathfrak{g}_z} = 0.$$

Then

$$\mathcal{S}(X) = \mathcal{S}(X - Z) + \mathfrak{g}(\mathcal{S}(X) \otimes \chi).$$

For proof see Appendix B.

Corollary 2.2.16. *Let a unipotent group G act on a real algebraic manifold X. Let χ be a unitary character of G.*

Let $Z \subset X$ be a Zariski closed G-invariant subset. Suppose also that for any point $z \in Z$ the restriction $\chi|_{G_z}$ is non-trivial. Then

$$\mathcal{S}(X) \otimes \chi = \mathcal{S}(X - Z) \otimes \chi + \mathfrak{g}(\mathcal{S}(X) \otimes \chi),$$

where $\mathfrak{g}$ is the Lie algebra of G.

Proof. The action of G_z on $\mathrm{Sym}^k(CN_{Gz,z}^X) \otimes ((\Delta_G)|_{G_z}/\Delta_{G_z})$ is algebraic and hence if G is unipotent this action is unipotent and therefore if $(\chi)_{\mathfrak{g}_z} = 0$ then

$$(\chi \otimes \mathrm{Sym}^k(CN_{Gz,z}^X) \otimes ((\Delta_G)|_{G_z}/\Delta_{G_z}))_{\mathfrak{g}_z} = 0.$$

□

Remark 2.2.17. *Note that the statement that $\mathcal{S}(X) \otimes \chi = \overline{\mathcal{S}(X - Z) \otimes \chi + \mathfrak{g}(\mathcal{S}(X) \otimes \chi)}$ is equivalent to the statement that any G-invariant distribution on X which is supported on Z vanishes, which is a generalization of a result from* [KV96].

2.3. **Nuclear Fréchet spaces.**

A good exposition on nuclear Fréchet spaces can be found in Appendix A of [CHM00].

We will need the following well-known facts from the theory of nuclear Fréchet spaces.

Proposition 2.3.1 (see e.g. [CHM00], Appendix A)**.**
Let V be a nuclear Fréchet space and W be a closed subspace. Then both W and V/W are nuclear Fréchet spaces.

Proposition 2.3.2 (see e.g. [CHM00], Appendix A)**.**
Let $0 \to V \to W \to U \to 0$ be an exact sequence of nuclear Fréchet spaces. Suppose that the embedding $V \to W$ is closed. Let L be a nuclear Fréchet space. Then the sequence $0 \to V\widehat{\otimes}L \to W\widehat{\otimes}L \to U\widehat{\otimes}L \to 0$ is exact and the embedding $V\widehat{\otimes}L \to W\widehat{\otimes}L$ is closed.

Corollary 2.3.3.
Let $V \to W$ be onto map between nuclear Fréchet spaces and L be a nuclear Fréchet space. Then the map $V\widehat{\otimes}L \to W\widehat{\otimes}L$ is onto.

Corollary 2.3.4. *Let $\phi_i : V_i \to W_i$ $i = 1, 2$ be onto maps between nuclear Fréchet spaces. Then the map $\phi_1\widehat{\otimes}\phi_2 : V_1\widehat{\otimes}V_2 \to W_1\widehat{\otimes}W_2$ is onto.*

Proposition 2.3.5 (see e.g. [AG10], Corollary 2.6.2)**.**
Let M be a Nash manifold. Then $\mathcal{S}(M)$ is a nuclear Fréchet space.

Proposition 2.3.6 (Schwartz Kernel Theorem, see e.g. [AG10], Corollary 2.6.3)**.**
Let M_i, $i = 1, 2$ be Nash manifolds Then

$$\mathcal{S}(M_1 \times M_2) = \mathcal{S}(M_1)\widehat{\otimes}\mathcal{S}(M_2).$$

Definition 2.3.7. *By a subspace of a topological vector space V we mean a linear subspace $L \subset V$ equipped with a topology such that the embedding $L \subset V$ is continuous.*

Note that by Banach open map theorem if L and V are nuclear Fréchet spaces and L is closed in V then the topology of L is the induced topology from V.

By an image of a continuous linear map between topological vector spaces we mean the image equipped with the quotient topology. Similarly for a continuous linear map between topological vector spaces $\phi : V_1 \to V_2$ and a subspace $L \subset V_1$ we the image $\phi(L)$ to be equipped with the quotient topology.

Similarly a ***sum*** *of two subspaces will be considered with the quotient topology of the direct sum.*

Remark 2.3.8. *Note that by Proposition 2.3.1, sum of nuclear Fréchet spaces and image of a nuclear Fréchet space are nuclear Fréchet spaces.*

Note also the operations of taking sum of subspaces and image of subspace commute.

Finally note that if L and L' are two nuclear Fréchet subspaces of a complete locally convex topological vector space V which coincide as linear subspaces then they are the same. Indeed, by Banach open map theorem they are both the same as $L + L'$.

Notation 2.3.9. *Let V_i, $i = 1, 2$ be locally convex complete topological vector spaces. Let $L_i \subset V_i$ be subspaces. We denote by $\mathcal{M}^{V_1,V_2}_{L_1,L_2} : L_1 \widehat{\otimes} L_2 \to V_1 \widehat{\otimes} V_2$ the natural map.*

From Corollary 2.3.4 we obtain the following corollary.

Corollary 2.3.10. *Let V_i, $i = 1, 2$ be locally convex complete topological vector spaces. Let L_i, $i = 1, 2$ be nuclear Fréchet spaces. Let $\phi_i : L_i \to V_i$ be continuous linear maps. Then*

$$\mathrm{Im}(\phi_1 \widehat{\otimes} \phi_2) = \mathrm{Im}(\mathcal{M}^{V_1,V_2}_{\mathrm{Im}(\phi_1),\mathrm{Im}(\phi_2)}).$$

Notation 2.3.11. *Let M_i, $i = 1, 2$ be smooth manifolds. We denote by $\mathcal{M}_{M_1,M_2} : C^\infty(M_1) \widehat{\otimes} C^\infty(M_2) \to C^\infty(M_1 \times M_2)$ the product map. For two subspaces $L_i \subset C^\infty(M_i)$ we denote by $\mathcal{M}_{L_1,L_2} : L_1 \widehat{\otimes} L_2 \to C^\infty(M_1 \times M_2)$ the composition $\mathcal{M}_{M_1,M_2} \circ \mathcal{M}^{C^\infty(M_1),C^\infty(M_2)}_{L_1,L_2}$.*

3. Proof of the main result

3.1. Notation.

In this paper we let D be a semi-simple 2-dimensional algebra over $\mathbb{R}$, i.e. $D = \mathbb{C}$ or $D = \mathbb{R} \oplus \mathbb{R}$. Let $a \mapsto \overline{a}$ denote the non-trivial involution of D, i.e. complex conjugate or swap respectively. Let n be a natural number. Let $\psi : \mathbb{R} \to \mathbb{C}^\times$ be a nontrivial character. The following notation will be used throughout the body of the paper. In case when there is no ambiguity we will omit from the notations the n, the D or the ψ.

- Denote by $H^n(D)$ the space of hermitian matrices of size n with coefficients in D.
- Denote $S^n(D) := H(D) \cap GL_n(D)$.
- Denote by $\Delta^n_i : H^n(D) \to \mathbb{R}$ the main i-minor.
- Let $N^n(D) < GL_n(D)$ be the subgroup consisting of upper unipotent matrices.
- Let $\mathfrak{n}^n(D)$ denote the Lie algebra of $N^n(D)$.
- We define a character $\chi_\psi : N^n(D) \to \mathbb{C}^\times$ by $\chi_\psi(x) := \psi(\sum_{i=1}^{n-1}(x_{i,i+1} + \overline{x_{i,i+1}}))$.
- Let the group $N^n(D)$ act on $H^n(D)$ by $x \mapsto \overline{u}^t x u$.
- Fix a symmetric $\mathbb{R}$-bilinear form B^n_D on $H^n(D)$ by $B(x, y) := \mathrm{Tr}_{\mathbb{R}}(xwyw)$, where $w := w_n$ is the longest element in the Weyl group of GL_n.
- Denote by $A^n < GL_n(\mathbb{R})$ the subgroup of diagonal matrices. We will also view A^n as a subset of $S^n(D)$.
- Define $\Omega^{n,\psi}_D : \mathcal{S}^{\det,\mathbb{R}^\times}(S^n(D)) \to C^\infty(A^n)$ by

$$\Omega^{n,\psi}_D(\Psi)(a) := \int_N \Psi(\overline{u}^t a u)\chi(u)du.$$

Here, du is the standard Haar measure on N.

For proof that the integral converges absolutely, depends smoothly on a and defines a continuous map $\mathcal{S}^{\det}(S^n(D)) \to C^\infty(A^n)$ see Proposition 3.1.1 below. By Remark 2.2.10, $\Omega^{n,\psi}_D$ defines in particular a continuous map $\mathcal{S}(H^n(D)) \to C^\infty(A^n)$.

- Define a character $\eta_D : \mathbb{R}^\times \to \{\pm 1\}$ by $\eta_D = 1$ if $D = \mathbb{R} \oplus \mathbb{R}$ and $\eta_D = \mathrm{sign}$ if $D = \mathbb{C}$.
- Define $\sigma : H^n(D) \to \mathbb{R}$ by $\sigma(x) := \prod_{i=1}^{n-1} \Delta_i^n(x)$.
- Define $\widetilde{\Omega}_D^{n,\psi} : \mathcal{S}^{\det,\mathbb{R}^\times}(S^n(D)) \to C^\infty(A^n)$ by

$$\widetilde{\Omega}_D^n(\Psi)(a) := \eta(\sigma(a))|\sigma(a)|\Omega(\Psi)(a)$$

Proposition 3.1.1.
The integral $\Omega_D^{n,\psi}$ converges absolutely and defines a continuous map

$$\Omega_D^{n,\psi} : \mathcal{S}^{\det}(S^n(D)) \to C^\infty(A^n).$$

Proof.
Consider the map $\beta : H^n(D) \to \mathbb{R}^n$ defined by $\beta = (\Delta_1, ..., \Delta_n)$. Consider A^n to be embedded in $\mathbb{R}^n$ by $(t_1, ...t_n) \mapsto (t_1, t_1t_2, ..., t_1t_2...t_n)$. Let $V := \beta^{-1}(A^n) \subset H^n(D)$. Let $p_n : \mathbb{R}^n \to \mathbb{R}$ denote the projection on the last coordinate. Note that the action map defines an isomorphism $N^n(D) \times A^n \to V$. Let $\alpha : V \to N^n(D)$ denote the projection. Let $\mathfrak{X} \in \mathcal{S}^{Id}(V)$ be defined by $\mathfrak{X}(v) := \chi(\alpha(v))$. Define $\Omega' : \mathcal{S}^{\beta,A}(V) \to \mathcal{S}^{Id}(A)$ by $\Omega'(f) := \beta_*(\mathfrak{X}f)$. Now, $\Omega_D^{n,\psi}$ is given by the following composition

$$\mathcal{S}^{\det,\mathbb{R}^\times}(S) \subset \mathcal{S}^{\beta,p_n^{-1}(\mathbb{R}^\times)}(S) \subset \mathcal{S}^{\beta,A}(V) \xrightarrow{\Omega'} \mathcal{S}^{Id}(A) \subset C^\infty(A).$$

□

The main theorem (Theorem A) can be reformulated now in the following way:

Theorem 3.1.2.
(i) $\widetilde{\Omega}^n_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}^n_{\mathbb{C}}(\mathcal{S}(H(\mathbb{C})))$.
(ii) $\widetilde{\Omega}^n_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(S(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}^n_{\mathbb{C}}(\mathcal{S}(S(\mathbb{C})))$.

3.2. **Main ingredients.**
In this subsection we list three main ingredients of the proof of the main theorem.

3.2.1. *Intermediate Kloosterman Integrals.* First, let us introduce the intermediate Kloosterman integrals and some related notation.

Notation 3.2.1.

- *Let $O_i^n(D) \subset H^n(D)$ be the subset of matrices with $\Delta_i^n \neq 0$.*
- *Let $U^n(D) := \bigcup_{i=1}^{n-1} O_i(D)$ and $Z^n(D) := H^n(D) - U^n(D)$.*
- *Denote by $N_i^n(D) < N^n(D)$ the subgroup defined by*

$$N_i^n(D) := \left\{ \begin{pmatrix} Id_i & * \\ 0 & Id_{n-i} \end{pmatrix} \right\}.$$

- *Define $\Omega_{D,i}^{n,\psi} : \mathcal{S}^{\Delta_i}(O_i^n(D)) \to \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(S^i(D) \times H^{n-i}(D))$, where $S^i \times H^{n-i}$ is considered as a subspace of H^n, in the following way*

$$\Omega_{D,i}^{n,\psi}(\Psi)(a) := \int_{N_i^n} \Psi(\overline{u}^t a u)\chi(u)du.$$

  Here, du is the standard Haar measure on N_i^n.
  For proof that the integral converges absolutely, depends smoothly on a and defines a continuous map $\mathcal{S}^{\Delta_i}(O_i^n(D)) \to \mathcal{S}^{\Delta_i}(S^i(D) \times H^{n-i}(D))$ see Proposition 3.2.2 below.
- *Define $\widetilde{\Omega}_{D,i}^{n,\psi} : \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(O_i^n(D)) \to \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(S^i(D) \times H^{n-i}(D))$, in the following way*

$$\widetilde{\Omega}_{D,i}^{n,\psi}(\Psi)(a) := \eta(\Delta_i^n(a))^{n-i}|\Delta_i^n(a)|^{n-i}\Omega_{D,i}^n.$$

- *We define $\Omega_D^{n_1,...,n_k,\psi} : \mathcal{S}^{\det\times...\times\det}(S^{n_1}(D) \times ... \times S^{n_k}(D)) \to C^\infty(A^{n_1} \times ... \times A^{n_k})$ in a similar way to $\Omega_D^{n,\psi}$. Analogously we define $\widetilde{\Omega}_D^{n_1,...,n_k,\psi}$.*

As before, in case when there is no ambiguity we will omit from the notations the n, the D or the ψ. In particular, we will omit the D and the ψ till the end of this subsubsection.

Proposition 3.2.2. *The integral Ω_i^n converges absolutely and defines a continuous map*

$$\Omega_i^n : \mathcal{S}^{\Delta_i}(O_i^n) \to \mathcal{S}^{\Delta_i}(S^i \times H^{n-i}).$$

Proof. Consider $S^i \times H^{n-i}$ as a subset in H^n. Denote it by B. Consider the action map $N_i \times B \to H$. Note that it is an open embedding and its image is O_i. We consider the standard Haar measures on B and N_i, and their multiplication on O_i. Consider the projections: $\alpha_i : O_i \to N_i$ and $\beta_i : O_i \to B$. Let $\mathfrak{X}_i \in \mathcal{S}^{Id}(O_i)$ be defined by $\mathfrak{X}_i(v) := \chi(\alpha_i(v))$. Consider $(\beta_i)_* : \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(O_i) \to \mathcal{S}^{\Delta_i,\mathbb{R}^\times}(B)$. Now, $\Omega_i(f) = (\beta_i)_*(\mathfrak{X}_i f)$. □

Proposition 3.2.3.
(i) The map $\widetilde{\Omega}_i^n$ defines an onto map $\mathcal{S}(O_i^n) \to \mathcal{S}(S^i \times H^{n-i})$.
(ii) $\widetilde{\Omega}^n = \widetilde{\Omega}^{i,n-i} \circ \widetilde{\Omega}_i^n$.

Proof. (i) follows from Property 2.2.6, since the map β_i from the proof of Proposition 3.2.2 is a surjective submersion.
(ii) is straightforward. □

Proposition 3.2.4. $\widetilde{\Omega}^{m,n}(\mathcal{S}(S^m \times H^n)) = \operatorname{Im} \mathcal{M}_{\widetilde{\Omega}^m(\mathcal{S}(S^m)),\widetilde{\Omega}^n(\mathcal{S}(H^n))}$. *For the definition of $\mathcal{M}$ see Notation 2.3.11.*

Proof. Let $I : \mathcal{S}(S^m)\widehat{\otimes}\mathcal{S}(H^n) \cong \mathcal{S}(S^m \times H^n)$ denote the isomorphism given by the Schwartz kernel theorem (see Property 2.3.6).

Clearly $\widetilde{\Omega}^{m,n}(\mathcal{S}(S^m \times H^n)) = \operatorname{Im}(\widetilde{\Omega}^{m,n} \circ I)$.

Note that

$$\widetilde{\Omega}^{m,n} \circ I = \mathcal{M}_{A_m,A_n} \circ \widetilde{\Omega}^m|_{\mathcal{S}(S^m)}\widehat{\otimes}\widetilde{\Omega}^n|_{\mathcal{S}(H^n)}.$$

By Corollary 2.3.10,

$$\operatorname{Im}(\widetilde{\Omega}^m|_{\mathcal{S}(S^m)}\widehat{\otimes}\widetilde{\Omega}^n|_{\mathcal{S}(H^n)}) = \operatorname{Im} \mathcal{M}^{C^\infty(A^m),C^\infty(A^n)}_{\widetilde{\Omega}^m(\mathcal{S}(S^m)),\widetilde{\Omega}^n(\mathcal{S}(H^n))}$$

Now,

$$\widetilde{\Omega}^{m,n}(\mathcal{S}(S^m \times H^n)) = \operatorname{Im}(\mathcal{M}_{A_m,A_n} \circ \widetilde{\Omega}^m|_{\mathcal{S}(S^m)}\widehat{\otimes}\widetilde{\Omega}^n|_{\mathcal{S}(H^n)}) =$$
$$= \operatorname{Im}(\mathcal{M}_{A_m,A_n} \circ \mathcal{M}^{C^\infty(A^m),C^\infty(A^n)}_{\widetilde{\Omega}^m(\mathcal{S}(S^m)),\widetilde{\Omega}^n(\mathcal{S}(H^n))}) = \operatorname{Im} \mathcal{M}_{\widetilde{\Omega}^m(\mathcal{S}(S^m)),\widetilde{\Omega}^n(\mathcal{S}(H^n))}.$$

□

From the last two propositions we obtain the following corollary.

Corollary 3.2.5. $\widetilde{\Omega}^n(\mathcal{S}(O_i^n)) = \operatorname{Im} \mathcal{M}_{\widetilde{\Omega}^{n-i}(\mathcal{S}(S^{n-i})),\widetilde{\Omega}^i(\mathcal{S}(H^i))}$.

3.2.2. *Inversion Formula.*

Theorem 3.2.6 (Jacquet)**.**

$$\widetilde{\Omega}_D^{\overline{\psi}}(\mathcal{F}(f))(diag(a_1,...,a_n)) =$$
$$= c(\psi, D)^{n(n-1)/2} \int ... \int \widetilde{\Omega}_D^{\psi}(f)(diag(p_1,..p_n))\psi(-\sum_{i=1}^{n} a_{n+1-i}p_i + \sum_{i=1}^{n-1} 1/(a_{n-i}p_i))dp_n...dp_1.$$

Here, $c(\psi, D)$ is a constant, we will discuss it in §§4.2. The integral here is just an iterated integral. In particular we mean that the integral converges as an iterated integral.

The proof is essentially the same as in the p-adic case (see [Jac03a, Section 7]). For the sake of completeness we repeat it in §4.

Corollary 3.2.7. *Let $\Psi \in \mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))$ and $\Psi \in \mathcal{S}(H(\mathbb{C}))$. Suppose that*

$$\widetilde{\Omega}^{\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Psi) = \widetilde{\Omega}^{\psi}_{\mathbb{C}}(\Phi).$$

Then

$$c(\psi, \mathbb{R}\oplus\mathbb{R})^{-n(n-1)/2}\widetilde{\Omega}^{\overline{\psi}}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{F}(\Psi)) = c(\psi, \mathbb{C})^{-n(n-1)/2}\widetilde{\Omega}^{\overline{\psi}}_{\mathbb{C}}(\mathcal{F}(\Phi)).$$

Later we will see that $c(\psi, \mathbb{R}\oplus\mathbb{R}) = 1$.

3.2.3. *Key Lemma.*

Lemma 3.2.8 (Key Lemma)**.** *Consider the actions of N and $\mathfrak{n}$ on $\mathcal{S}(H)$ to be the standard actions twisted by χ. Then*

$$\mathcal{S}(H) = \mathcal{S}(U) + \mathcal{F}(\mathcal{S}(U)) + \mathfrak{n}\mathcal{S}(H).$$

For proof see §5.

3.3. **Proof of the main result.**

We prove Theorem 3.1.2 by induction. The base $n = 1$ is obvious. Thus, from now on we assume that $n \geq 2$ and that Theorem 3.1.2 holds for all dimensions smaller than n.

Proposition 3.3.1.

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(O_i(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(O_i(\mathbb{C}))).$$

Proof. Follows from Corollary 3.2.5 and the induction hypothesis. □

Corollary 3.3.2.

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(U(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(U(\mathbb{C}))).$$

Proof. Follows from the the previous proposition and partition of unity (property 2.2.4). □

Corollary 3.3.3. *Part (i) of Theorem 3.1.2 holds. Namely, $\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(H(\mathbb{C})))$.*

Proof. By the previous Corollary and the inversion formula (see Corollary 3.2.7),

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{F}(\mathcal{S}(U(\mathbb{R}\oplus\mathbb{R})))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{F}(\mathcal{S}(U(\mathbb{C})))).$$

Clearly, $\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathfrak{n}\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathfrak{n}\mathcal{S}(H(\mathbb{C}))) = 0$. Hence, by Remark 2.3.8

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(U(\mathbb{R}\oplus\mathbb{R})) + \mathcal{F}(\mathcal{S}(U(\mathbb{R}\oplus\mathbb{R}))) + \mathfrak{n}\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(U(\mathbb{C})) + \mathcal{F}(\mathcal{S}(U(\mathbb{C}))) + \mathfrak{n}\mathcal{S}(H(\mathbb{C}))),$$

where we again consider the actions of N and $\mathfrak{n}$ on $\mathcal{S}(H)$ to be twisted by χ. Therefore, by the Key Lemma

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(H(\mathbb{C}))).$$

□

It remains to prove part (ii) of Theorem 3.1.2.

Proof of part (ii) of Theorem 3.1.2. By Property 2.2.7,

$$\mathcal{S}(S(\mathbb{R}\oplus\mathbb{R})) = \mathcal{S}(\mathbb{R}^{\times})\mathcal{S}(S(\mathbb{R}\oplus\mathbb{R})),$$

and hence

$$\mathcal{S}(S(\mathbb{R}\oplus\mathbb{R})) = \mathcal{S}(\mathbb{R}^{\times})\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R})),$$

where the action of $\mathcal{S}(\mathbb{R}^{\times})$ on $\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))$ is given via $\det : H(\mathbb{R}\oplus\mathbb{R}) \to \mathbb{R}$.

Hence

$$\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(S(\mathbb{R}\oplus\mathbb{R}))) = \mathcal{S}(\mathbb{R}^{\times})\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))).$$

By part (i)

$$\mathcal{S}(\mathbb{R}^{\times})\widetilde{\Omega}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{S}(H(\mathbb{R}\oplus\mathbb{R}))) = \mathcal{S}(\mathbb{R}^{\times})\widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(H(\mathbb{C}))).$$

As before,

$$\mathcal{S}(\mathbb{R}^{\times})\widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(H(\mathbb{C}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(\mathbb{R}^{\times})\mathcal{S}(H(\mathbb{C}))) = \widetilde{\Omega}_{\mathbb{C}}(\mathcal{S}(S(\mathbb{C}))).$$

□

Remark 3.3.4. *One can give an alternative proof, that does not use Dixmier - Malliavin theorem (Property 2.2.7), in the following way. Define maps* $\widetilde{\Omega}' : \mathcal{S}(H \times \mathbb{R}^\times) \to C^\infty(A \times \mathbb{R}^\times)$ *similarly to* $\widetilde{\Omega}$, *and not involving the second coordinate. From (i), using §2.3, we get that* $\mathrm{Im}\, \widetilde{\Omega}'_{\mathbb{C}} = \mathrm{Im}\, \widetilde{\Omega}'_{\mathbb{R}\oplus\mathbb{R}}$. *Using the graph of* $\det$ *we can identify* S *with a closed subset of* $H \times \mathbb{R}^\times$ *and* A *with a closed subset of* $A \times \mathbb{R}^\times$. *By Property 2.2.5, the restriction map* $\mathcal{S}(H \times \mathbb{R}^\times) \to \mathcal{S}(S)$ *is onto and hence* $\widetilde{\Omega}(\mathcal{S}(S)) = \mathrm{Im}\, res \circ \widetilde{\Omega}'$, *where* $res : C^\infty(A \times \mathbb{R}^\times) \to C^\infty(A)$ *is the restriction. This implies (ii).*

In fact, this alternative proof of (ii) is obtained from the previous proof by replacing Property 2.2.7 with its weaker version that states (in the notation of property 2.2.7) that the map $\mathcal{S}(M)\widehat{\otimes}\mathcal{S}(N) \to \mathcal{S}(M)$ *is onto. This is much simpler version since it follows directly from Property 2.2.5 and Proposition 2.3.6.*

4. Proof of the inversion formula

In this section we adapt the proof of Theorem 3.2.6 given in [Jac03a] to the Archimedean case. The proof is by induction. The induction step is based on analogous formula for the intermediate Kloostermann integral which is based on the Weil formula.

In §§4.1 we give notations for various Fourier transforms on H. In §§4.2 we recall the Weil formula and consider its special case which is relevant for us. In §§4.3 we introduce the Jacquet transform and the intermediate Jacquet transform which appears on the right hand side of the inversion formulas. In §§4.4 we prove the intermediate inversion formula. In §§4.5 we prove the inversion formula.

4.1. Fourier transform.

- We denote by $\mathcal{F}' := \mathcal{F}'_{H^n} : \mathcal{S}(H^n) \to \mathcal{S}(H^n)$ the Fourier transform w.r.t. the trace form (and the character ψ).
- Note that $\mathcal{F}_{H^n} = ad(w) \circ \mathcal{F}'_{H^n} = \mathcal{F}'_{H^n} \circ ad(w)$.
- We denote by $\mathcal{F}'_{H^i \times H^{n-i}} : \mathcal{S}(H^n) \to \mathcal{S}(H^n)$ the partial Fourier transform w.r.t. the trace form on $H^i \times H^{n-i}$.
- We denote by $(H^i \times H^{n-i})^{\perp'} \subset H^n$ the orthogonal compliment to $H^i \times H^{n-i}$ w.r.t. the trace form.
- We denote by $\mathcal{F}'_{(H^i \times H^{n-i})^{\perp'}} : \mathcal{S}(H^n) \to \mathcal{S}(H^n)$ the partial Fourier transform w.r.t. the trace form on $(H^i \times H^{n-i})^{\perp'}$.
- Note that $\mathcal{F}'_{H^n} = \mathcal{F}'_{(H^i \times H^{n-i})^{\perp'}} \circ \mathcal{F}'_{H^i \times H^{n-i}} = \mathcal{F}'_{H^i \times H^{n-i}} \circ \mathcal{F}'_{(H^i \times H^{n-i})^{\perp'}}$.

4.2. The Weil formula.

Recall the one dimensional Weil formula:

Proposition 4.2.1. *Let* $a \in \mathbb{R}^\times$. *Consider the function* $\xi : D \to \mathbb{R}$ *defined by* $\xi(x) = \psi(ax\bar{x})$ *as a distribution on* D. *Then* $\mathcal{F}^*(\xi) = \zeta$, *where* ζ *is a distribution defined by the function* $\zeta(x) = |a|^{-1}\eta_D(a)c(D,\psi)\psi(-x\bar{x}/a)$.

One can take this as a definition of the constant $c(D,\psi)$.

The following proposition is proven by a straightforward computation.

Proposition 4.2.2.
(i) $c(\mathbb{R} \oplus \mathbb{R}, \psi) = 1$
(ii) $c(\mathbb{C}, \psi)^2 = -1$
(iii) $c(\mathbb{C}, \psi)c(\mathbb{C}, \overline{\psi}) = 1$

Proposition 4.2.1 gives us the following corollary.

Corollary 4.2.3. *Let* V *be a free module over* D *equipped with a volume form. We have a natural Fourier transform* $\mathcal{F}^* : \mathcal{S}^*(V) \to \mathcal{S}^*(V^*)$. *Let* Q *be a hermitian form on* V. *Consider the function* $\xi : V \to \mathbb{R}$ *defined by* $\xi(v) = \psi(Q(v))$ *as a distribution on* V. *Let* Q^{-1} *be a hermitian norm on* V^* *which is the inverse of* Q. *Let* $\det(Q)$ *be the determinant of* Q *with respect to the volume form on* V. *Let* ζ *be a distribution defined by the function*

$$\zeta(x) = |\det(Q)|^{-1}(\eta_D(\det(Q)c(D,\psi))^{\dim V}\psi(-Q^{-1}(x)).$$

Then $\mathcal{F}^*(\xi) = \zeta$.

Corollary 4.2.4. *Let $(A,B) \in S^i \times S^{n-i}$. Consider the function $\xi : (H^i \times H^{n-i})^{\perp'} \to \mathbb{R}$ defined by $\xi\left[\begin{pmatrix} 0 & \bar{u}^t \\ u & 0 \end{pmatrix}\right] = \psi(BuA\bar{u}^t)$ as a distribution on V. Consider also the function $\zeta : (H^i \times H^{n-i})^{\perp'} \to \mathbb{R}$ defined by*

$$\zeta\left[\begin{pmatrix} 0 & \bar{u}^t \\ u & 0 \end{pmatrix}\right] = (\eta(\det A)/|\det A|)^{n-i}(\eta(\det B)/|\det B|)^i c(D,\psi)^{(n-i)i}\psi(B^{-1}\bar{u}^t A^{-1}u)$$

as a distribution on V.

Then $(\mathcal{F}'_{(H^i \times H^{n-i})^{\perp'}})^*(\xi) = \zeta$.

4.3. **Jacquet transform.**

Definition 4.3.1. *Let $0 \le i \le n$.*

- *We define $\mathcal{J}'_{i,n-i} : C^\infty(S^i \times S^{n-i}) \to C^\infty(S^i \times S^{n-i})$ by $\mathcal{J}'_{i,n-i}(f)(A,B) = f(A,B)\psi(wB^{-1}w\varepsilon A^{-1}\varepsilon^t)$. Here ε is the matrix with $n-i$ rows and i columns whose first row is the row $(0,0,...,0,1)$ and all other rows are zero.*
- *We define $\mathcal{T}_{i,n-i} : C^\infty(S^i \times S^{n-i}) \to C^\infty(S^{n-i} \times S^i)$ by $\mathcal{T}_{i,n-i}(f)(A,B) = f(B,A)$.*
- *We denote by $\mathfrak{J}_{i,n-i}$ the space $\mathcal{S}^{\Delta_i,\mathbb{R}^\times}(S^i \times H^{n-i}) \cap \mathcal{F}^{-1}_{H^{n-i}}(\mathcal{T}^{-1}_{i,n-i}(\mathcal{J}'^{-1}_{i,n-i}(\mathcal{S}^{\Delta_{n-i},\mathbb{R}^\times}(S^{n-i} \times H^i))))$.*
- *We define the partial Jacquet transform $\mathcal{J}_{i,n-i} : \mathfrak{J}_{i,n-i} \to \mathcal{S}^{\Delta_{n-i},\mathbb{R}^\times}(S^{n-i} \times H^i))$ by*
$$\mathcal{J}_{i,n-i} := \mathcal{F}_{H^i} \circ \mathcal{T}_{i,n-i} \circ \mathcal{J}'_{i,n-i} \circ \mathcal{F}_{H^{n-i}}|_{\mathfrak{J}_{i,n-i}}.$$
- *Denote by $\overline{A^n}$ the set of diagonal matrices in H.*
- *We denote $\mathcal{F}_n : \mathcal{S}^{\Delta_{n-1}}(\overline{A^n}) \to \mathcal{S}^{\Delta_{n-1}}(\overline{A^n})$ the Fourier transform w.r.t. the last co-ordinate.*
- *We define*
$$\mathcal{J}^{(i)'}_n : \mathcal{S}^{\Delta_{n-1}}(\overline{A^n}) \to C^\infty(A^n)$$
by
$$\mathcal{J}^{(i)'}_n(f)(a_1,...,a_n) = f(a_1,...,a_{i-1},a_n,a_i..,a_{n-1})\psi(1/a_n a_{n-1}).$$
- *We define $\mathcal{J}^{(i)}_n : \mathcal{S}^{\Delta_{n-1}}(\overline{A^n}) \to C^\infty(A^n)$ by $\mathcal{J}^{(i)}_n = \mathcal{J}^{(i)'}_n \circ \mathcal{F}_n$ for $i<n$.*
- *We define inductively a sequence of subspaces $\mathfrak{J}^{[i]}_n \subset C^\infty(A^n)$ and operators $\mathcal{J}^{[i]}_n : \mathfrak{J}^{[i]}_n \to C^\infty(A^n)$ in the following way $\mathfrak{J}^{[1]}_n = \mathcal{S}^{\Delta_{n-1}}(\overline{A^n})$, $\mathcal{J}^{[1]}_n = \mathcal{F}_n$, $\mathfrak{J}^{[i]}_n = \mathcal{S}^{\Delta_{n-1}}(\overline{A^n}) \cap (\mathcal{J}^{(i)}_n)^{-1}(\mathfrak{J}^{[i-1]}_n)$ and $\mathcal{J}^{[i]}_n = \mathcal{J}^{[i-1]}_n \circ \mathcal{J}^{(n+1-i)}_n$.*
- *We define the Jacquet space $\mathfrak{J} := \mathfrak{J}_n$ to be $\mathfrak{J}^{[n]}_n$ and the Jacquet transform $\mathcal{J} := \mathcal{J}_n : \mathfrak{J} \to C^\infty(A^n)$ to be $\mathcal{J}^{[n]}_n$.*

4.4. **The partial inversion formula.**
In this subsection we prove an analog of Proposition 8 of [Jac03a], namely

Proposition 4.4.1.
(i) $\widetilde{\Omega}^\psi_i(\mathcal{S}(H)) \subset \mathfrak{J}_{i,n-i}$
(ii) $\mathcal{J}_{i,n-i} \circ \widetilde{\Omega}^\psi_i|_{\mathcal{S}(H)} = c(D,\psi)^{n(n-i)}\widetilde{\Omega}^{\bar{\psi}}_{n-i} \circ \mathcal{F}_H$

This proposition is equivalent to the following one

Proposition 4.4.2.
$$\mathcal{J}'_{i,n-i} \circ \mathcal{F}_{H^{n-i}} \circ \widetilde{\Omega}^\psi_{i,n-i}|_{\mathcal{S}(H)} = c(D,\psi)^{n(n-i)}\mathcal{T}^{-1}_{i,n-i} \circ (\mathcal{F}_{H^i})^{-1} \circ \widetilde{\Omega}^{\bar{\psi}}_{n-i} \circ \mathcal{F}.$$

For its proof we will need some auxiliary results.

Lemma 4.4.3. *Let $f \in \mathcal{S}(H)$ be a Schwartz function. Then*

$$\widetilde{\Omega}^\psi_i(f)(A,B) = \eta(\det(A))^{n-i}|\det(A)|^{-(n-i)}\int f\left[\begin{pmatrix} A & X \\ \bar{X}^t & B + X^t A^{-1} X \end{pmatrix}\right]\psi[\mathrm{Tr}(\varepsilon A^{-1}X) + \mathrm{Tr}(\bar{X}^t A^{-1}\varepsilon^t)]dX$$

The proof is straightforward.

Corollary 4.4.4. *Let $f \in \mathcal{S}(H)$ be a Schwartz function. Then*

$$\mathcal{F}_{H^{n-i}} \circ \widetilde{\Omega}_i^{\psi}(f)(A, w_{n-i}Cw_{n-i}) = \eta(\det(A))^{n-i}|\det(A)|^{-(n-i)}$$
$$\int f\left[\begin{pmatrix} A & X \\ \bar{X}^t & B \end{pmatrix}\right]\psi[\mathrm{Tr}(\varepsilon A^{-1}X) + \mathrm{Tr}(\bar{X}^t A^{-1}\varepsilon^t) + \mathrm{Tr}(CX^tA^{-1}X) - Tr(CB)]dXdB$$

Notation 4.4.5.
(i) Let $\xi_{A,B} \in \mathcal{S}^*(H)$ be the distribution defined by

$$\xi_{A,B}(f) = \mathcal{J}'_{i,n-i} \circ \mathcal{F}_{H^{n-i}} \circ \widetilde{\Omega}_i^{\psi}(f)(A, B).$$

(ii) Let $\zeta_{A,B} \in \mathcal{S}^*(H)$ be the distribution defined by

$$\zeta_{A,B}(f) = \mathcal{T}_{i,n-i}^{-1} \circ (\mathcal{F}_{H^i})^{-1} \circ \widetilde{\Omega}_{n-i}^{\bar{\psi}}(f)(A, B).$$

Proof of Proposition 4.4.2. We have to show that

$$\xi_{A,B} = c(D,\psi)^{n(n-i)}\mathcal{F}(\zeta_{A,B})$$

Let $f \in \mathcal{S}(H)$ be a Schwartz function. Denote $m := n - i$. By Corollary 4.4.4

$$\xi_{A,C}(f) = \eta(\det(A))^{n-i}|\det(A)|^{-(n-i)}\psi(w_{n-i}C^{-1}w_{n-i}\varepsilon A^{-1}\varepsilon^t)$$
$$\int f\left[\begin{pmatrix} A & X \\ \bar{X}^t & B \end{pmatrix}\right]\psi[\mathrm{Tr}(\varepsilon A^{-1}X)+\mathrm{Tr}(\bar{X}^tA^{-1}\varepsilon^t)+\mathrm{Tr}(w_{n-i}Cw_{n-i}X^tA^{-1}X)-Tr(w_{n-i}Cw_{n-i}B)]dXdB$$

and

$$\zeta_{A,C}(f) = \eta(\det(C))^i|\det(C)|^{-i}\times$$
$$\int f\left[\begin{pmatrix} C & X \\ \bar{X}^t & B \end{pmatrix}\right]\psi[-\mathrm{Tr}(\varepsilon C^{-1}X + \bar{X}^tC^{-1}\varepsilon^t + w_iAw_iX^tC^{-1}X - w_iAw_iB)]dXdB.$$

Therefore

$$ad(w_n)(\zeta_{A,C})(f) = \eta(\det(C))^i|\det(C)|^{-i}\times$$
$$\int f\left[\begin{pmatrix} B & X \\ \bar{X}^t & w_mCw_m \end{pmatrix}\right]\psi[-\mathrm{Tr}(\varepsilon C^{-1}w_m\bar{X}^tw_m + w_mXw_mC^{-1}\varepsilon^t + AXw_mC^{-1}w_m\bar{X}^t - AB)]dXdB.$$

Thus

$$\mathcal{F}'_{H^i\times H^m}(ad(w_n)(\zeta_{A,C}))(f) = \eta(\det(C))^i|\det(C)|^{-i}\times$$
$$\int f\left[\begin{pmatrix} A & X \\ \bar{X}^t & B \end{pmatrix}\right]\psi[-\mathrm{Tr}(\varepsilon C^{-1}w_m\bar{X}^tw_m + w_mXw_mC^{-1}\varepsilon^t + AXw_mC^{-1}w_m\bar{X}^t + w_mCw_mB)]dXdB.$$

Therefore by Corollary 4.2.4

$$\mathcal{F}'_{(H^i\times H^m)^{\perp'}}(\mathcal{F}'_{H^i\times H^m}(ad(w_n)(\zeta_{A,C})))(f) = c(D,\psi)^{n(n-i)}\xi_{A,C}(f).$$

□

4.5. **Proof of the inversion formula.**
The inversion formula (Theorem 3.2.6) is equivalent to the following theorem.

Theorem 4.5.1.
(i) $\widetilde{\Omega}^{n,\psi}(\mathcal{S}(H)) \subset \mathfrak{J}$.
(ii) $\mathcal{J} \circ \widetilde{\Omega}^{n,\psi}|_{\mathcal{S}(H)} = c(D,\psi)^{n(n-1)/2}\widetilde{\Omega}^{n,\bar{\psi}} \circ \mathcal{F}_H$.

The proof is by induction. We will need the following straightforward lemma.

Lemma 4.5.2. *The induction hypotheses implies that*
(i) $\widetilde{\Omega}^{1,n-1,\psi}(\mathcal{S}^{\Delta_1}(S^1 \times H^{n-1})) \subset \mathfrak{J}_n^{[n-1]}$
(ii)

$$\widetilde{\Omega}^{1,n-1,\bar{\psi}} \circ \mathcal{F}_{H^{n-1},\psi} = c(D,\psi)^{(n-1)(n-2)/2} \mathcal{J}_n^{[n-1]} \widetilde{\Omega}^{1,n-1,\psi}|_{\mathcal{S}^{\Delta_1}(S^1 \times H^{n-1})}$$

Proof of Theorem 4.5.1. First let us prove (i). It is easy to see that

$$\widetilde{\Omega}^{1,n-1,\psi}|_{\mathcal{S}^{\Delta_1}(S^1\times H^{n-1})} \circ \mathcal{T}_{n-1,1} \circ \mathcal{J}'_{n-1,1}|_{\mathcal{F}_{H^1}(\mathfrak{J}_{n-1,1})} = \mathcal{J}_n^{(i)'} \circ \widetilde{\Omega}^{1,n-1,\psi}|_{\mathcal{F}_{H^1}(\mathfrak{J}_{n-1,1})} \tag{1}$$

This implies that

$$\widetilde{\Omega}^{1,n-1,\psi}|_{\mathcal{S}^{\Delta_1}(S^1\times H^{n-1})} \circ \mathcal{T}_{n-1,1} \circ \mathcal{J}'_{n-1,1} \circ \mathcal{F}_{H^1} \circ \widetilde{\Omega}^{n,\psi}_{n-1}|_{\mathcal{S}(H)} = \mathcal{J}_n^{(i)'} \circ \mathcal{F}_n \circ \widetilde{\Omega}^{1,n-1,\psi} \circ \widetilde{\Omega}^{n,\psi}_{n-1}|_{\mathcal{S}(H)} \tag{2}$$

By Proposition 3.2.3 this implies

$$\widetilde{\Omega}^{1,n-1,\psi}|_{\mathcal{S}^{\Delta_1}(S^1\times H^{n-1})} \circ \mathcal{T}_{n-1,1} \circ \mathcal{J}'_{n-1,1} \circ \mathcal{F}_{H^1} \circ \widetilde{\Omega}^{n,\psi}_{n-1}|_{\mathcal{S}(H)} = \mathcal{J}_n^{(i)'} \circ \mathcal{F}_n \circ \widetilde{\Omega}^{n,\psi}|_{\mathcal{S}(H)} \tag{3}$$

This together with Lemma 4.5.2 implies (i).

Now let us prove (ii). By Propositions 3.2.3 and 4.4.1 we have

$$\begin{aligned}\widetilde{\Omega}^{n,\bar{\psi}} \circ \mathcal{F}_H = \widetilde{\Omega}^{1,n-1,\bar{\psi}} \circ \widetilde{\Omega}^{n,\bar{\psi}}_1 \circ \mathcal{F}_H = c(D,\psi)^{(n-1)} \widetilde{\Omega}^{1,n-1,\bar{\psi}} \circ \mathcal{J}_{n-1,1} \circ \widetilde{\Omega}^{n,\psi}_{n-1}|_{\mathcal{S}(H)} = \\ = c(D,\psi)^{(n-1)} \widetilde{\Omega}^{1,n-1,\bar{\psi}} \circ \mathcal{F}_{H^{n-1}} \circ \mathcal{T}_{n-1,1} \circ \mathcal{J}'_{n-1,1} \circ \mathcal{F}_{H^1} \circ \widetilde{\Omega}^{n,\psi}_{n-1}|_{\mathcal{S}(H)}\end{aligned} \tag{4}$$

(ii) follows now from (3), (4), and Lemma 4.5.2. ☐

5. Proof of the Key Lemma

We will use the following notation and lemma.

Notation 5.0.1. *Denote*

$$Z' := \left\{ \begin{pmatrix} 0 & \cdots & 0 & a \\ \vdots & \iddots & \iddots & * \\ 0 & a & \iddots & \vdots \\ a & * & \cdots & * \end{pmatrix} \text{ s.t. } a \in \mathbb{R} \right\} \cap H =$$

$$= \{x \in Z | x_{ij} = 0 \text{ for } i+j < n+1 \text{ and } x_{i,n+1-i} = x_{j,n+1-j} \in \mathbb{R} \text{ for any } 1 \le i,j \le n\} \subset Z$$

Denote also $U' := H - Z'$.

Notation 5.0.2. *We call a matrix* $x \in H$ *relevant if* $\chi|_{N_x} \equiv 1$*, and irrelevant otherwise.*

Lemma 5.0.3 ([Jac03a], §3, §5)**.** *Every relevant orbit in* $H^n(D)$ *has a unique representative of the form*

$$\begin{pmatrix} a_1 w_{m_1} & 0 & \cdots & 0 \\ 0 & a_2 w_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n w_{m_n} \end{pmatrix} \tag{5}$$

where $m_1 + ... + m_j = n$*,* $a_1, ..., a_j \in \mathbb{R}$*, and if* $\det(g) = 0$ *then* $\Delta_{n-1}(g) \neq 0$*.*

For the sake of completeness we will repeat the proof here.

Proof. Step 1. Proof for $S^n(\mathbb{R} \oplus \mathbb{R})$
Let W_n denote the group of permutation matrices. By Bruhat decomposition, every orbit has a unique representative of the form wa with $w \in W_n$ and $a \in A^n$. If this element is relevant, then for every pair of positive roots (α_1, α_2) such that $w\alpha_2 = -\alpha_1$, and for $u_i \in N_{\alpha_i}(\mathbb{R})$ (where N_{α_i} denotes the one-dimensional subgroup of N corresponding to α_i) we have

$$u_1^t w a u_2 = wa \Rightarrow \chi(u_1, u_2) = 0. \tag{6}$$

This condition implies that α_1 is simple if and only if α_2 is simple. Thus w and its inverse have the property that if they change a simple root to a negative one, then they change it to the opposite of a simple root. Let S be the set of simple roots α such that $w\alpha$ is negative. Then S is also the set of simple roots α such that $w^{-1}\alpha$ is negative and $wS = S$. Let M be the standard Levi subgroup determined by S. Thus S is the set of simple roots of M for the torus A, w is the longest element of the Weyl group of M, and $w^2 = 1$. This being so, if α_2 is simple, then condition (6) implies $\alpha_2(a) = 1$. Thus a is in the center of M. Hence wa is of the form (5).

Step 2. Proof for $S^n(\mathbb{C})$.

Every orbit has a unique representative of the form wa with $w \in W_n$, and diagonal $a \in GL_n(\mathbb{C})$ (for proof see e.g. [Spr85, Lemma 4.1(i)], for the involution $g \mapsto w_n \overline{g}^{-t} w_n$, where $w_n \in W_n$ denotes the longest element). Since $wa \in S$, we have $w = w^t$ and hence $w^2 = 1$ and $waw = \overline{a}$.

Suppose that α is a simple root such that $w\alpha = -\beta$ where β is positive. For $u_\alpha \in N_\alpha$, define

$$u_\beta := w\overline{a}^{-1}\overline{u}_\alpha^{-t}\overline{a}w \in N_\beta.$$

Then

$$\overline{u}_\beta^t wau_\alpha = wa = \overline{u}_\alpha^t wau_\beta.$$

There exists an element $u_{\alpha+\beta} \in N_{\alpha+\beta}$ (i.e. $u_{\alpha+\beta} = 1$ if $\alpha + \beta$ is not a root) such that $u := u_{\alpha+\beta}u_\alpha u_\beta$ satisfies $\overline{u}^t wau = wa$. If wa is relevant, this relation implies $\chi(u_\alpha u_\beta) = 1$.

Thus β is simple. Since $w^2 = 1$, we see that, as before, there is a standard Levi subgroup M such that w is the longest element in its Weyl group, and $a \in Z(M) \cap A^n$.

Step 3. Proof for $H^n(D) - S^n(D)$.

Let $s \in H^n(D)$ with $\det(s) = 0$ be relevant. Then $s = u^t wb$ with $u \in N(D)$, $w \in W_n$ and b upper triangular. If a column of s of index $i < n$ would be zero, then the row with index i would also be zero, and hence s would be irrelevant. Hence $b_{1,1} \neq 0$ and acting on s by $N(D)$ we can bring b to the form $b = \begin{pmatrix} b' & 0 \\ 0 & 0 \end{pmatrix}$, where b' is diagonal and invertible. In particular, the last row of b is zero. We may replace s by $wb\overline{u}^{-1}$. The last row of $b\overline{u}^{-1}$ is again zero. Since the rows of $wb\overline{u}^{-1}$ with index less than n cannot be zero, w must have the form $w = \begin{pmatrix} w' & 0 \\ 0 & 0 \end{pmatrix}$. The theorem follows now from the 2 previous cases. □

Since Z and Z' are N-invariant we obtain

Corollary 5.0.4. *Every relevant $x \in Z$ lies in Z'.*

Using Corollary 2.2.16 we obtain

Corollary 5.0.5. *Recall that we consider the action of N on $\mathcal{S}(H)$ to be the standard action twisted by χ. Then $\mathcal{S}(U') = \mathcal{S}(U) + \mathfrak{n}\mathcal{S}(U')$.*

Lemma 5.0.6. $Z' \not\supset Z'^{\perp}$.

Proof. For $n > 2$ this is obvious since $\dim Z' < \frac{n^2}{2} = \frac{\dim H}{2}$.

For $n = 2$, $\dim Z' = \frac{n^2}{2} = \frac{\dim H}{2}$. Hence it is enough to show that $Z' \neq (Z')^{\perp}$. Now

$$B\left(\begin{pmatrix} 0 & a \\ a & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ c & d \end{pmatrix}\right) = 2ac,$$

which is not identically 0. □

Corollary 5.0.7. $\mathcal{S}(H) = \mathcal{S}(U') + \mathcal{F}(\mathcal{S}(U'))$.

Proof. Follows from the previous lemma and Theorem 2.2.13. □

Proof of the Key Lemma (Lemma 3.2.8). By Corollaries 5.0.5 and 5.0.7,

$$\begin{aligned}\mathcal{S}(H) = \mathcal{S}(U') + \mathcal{F}(\mathcal{S}(U')) = \mathcal{S}(U) + \mathfrak{n}\mathcal{S}(U') + \mathcal{F}(\mathcal{S}(U) + \mathfrak{n}\mathcal{S}(U')) = \\ = \mathcal{S}(U) + \mathfrak{n}\mathcal{S}(U') + \mathcal{F}(\mathcal{S}(U)) + \mathfrak{n}\mathcal{F}(\mathcal{S}(U')) = \\ = \mathcal{S}(U) + \mathcal{F}(\mathcal{S}(U)) + \mathfrak{n}(\mathcal{S}(U') + \mathcal{F}(\mathcal{S}(U'))) \subset \mathcal{S}(U) + \mathcal{F}(\mathcal{S}(U)) + \mathfrak{n}(\mathcal{S}(H)).\end{aligned}$$

The opposite inclusion is obvious. □

6. Non-regular Kloostermann integrals

In this section we define Kloostermann integrals over relevant non-regular orbits. We prove that if two functions match then their non-regular Kloostermann integrals also equal, up to a matching factor. In the non-Archimedean case this was done in [Jac03b] and our proof follows the same lines.

We also deduce that if all regular Kloostermann integrals of a function vanish then all Kloostermann integrals of this function vanish. In the non-Archimedean case this was also proven in [Jac03b] and earlier and in a different way in [Jac98].

Recall that $g \in H^n(D)$ is called relevant if the character χ is trivial on the stabilizer $N(D)_g$ of g. For every relevant $g \in H^n(D)$ and every $\Psi \in \mathcal{S}^{\det,\mathbb{R}^\times}(S^n(D))$ we define

$$\Omega_D^{n,\psi}(\Psi, g) := \int_{N/N_g} \Psi(\overline{u}^t a u)\chi(u)du.$$

Recall the description of relevant orbits given in Lemma 5.0.3: every relevant orbit in $H^n(D)$ has a unique element of the form

$$g = \begin{pmatrix} a_1 w_{m_1} & 0 & \cdots & 0 \\ 0 & a_2 w_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n w_{m_n} \end{pmatrix}, \tag{7}$$

where $m_1 + ... + m_j = n$, $a_1, ..., a_j \in \mathbb{R}$, and if $\det(g) = 0$ then $\Delta_{n-1}(g) \neq 0$. In particular, $H^n(\mathbb{C})$ and $H^n(\mathbb{R} \oplus \mathbb{R})$ have the same set of representatives of regular orbits.

Notation 6.0.1. *We define the transfer factor γ to all g of the form (7) by*

- *For a scalar $a \in H^1$ we let $\gamma(a, \psi) := 1$*
- $\gamma(aw_n, \psi) = \gamma(-a^{-1}w_{n-1}, \overline{\psi})c(\mathbb{C}, \psi)^{n(n-1)/2}\operatorname{sign}(\det(-a^{-1}w_{n-1}))$
- *For $g = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ let $\gamma(g, \psi) := \gamma(x, \psi)\gamma(y, \psi)\operatorname{sign}(\det(x))^i$ where y is an $i \times i$ matrix.*

Note that this definition extends the definition of transfer factor γ that was given in the introduction for $g \in A^n$.

Remark 6.0.2. *Since $c(\mathbb{C}, \psi)^2 = -1$ and $c(\mathbb{C}, \psi)c(\mathbb{C}, \overline{\psi}) = 1$, we have $\gamma(aw_{n+8}, \psi) = \gamma(aw_n, \psi)$ and for $1 \leq n \leq 8$, $\gamma(aw_n, \psi)$ is determined by the sequence*

$$1,\ c(\mathbb{C}, \psi)\operatorname{sign}(-a),\ \operatorname{sign}(a),\ 1,\ -1,\ c(\mathbb{C}, \psi)\operatorname{sign}(-a),\ \operatorname{sign}(-a),\ 1.$$

In particular $\gamma(g, \psi)$ is always a fourth root of unity.

Theorem 6.0.3. *Let $\Phi \in \mathcal{S}^{\det,\mathbb{R}^\times}(H^n(\mathbb{R} \oplus \mathbb{R}))$ and $\Psi \in \mathcal{S}^{\det,\mathbb{R}^\times}(H^n(\mathbb{C}))$. Suppose that*

$$\Omega_{\mathbb{R}\oplus\mathbb{R}}^{n,\psi}(\Phi) = \gamma\Omega_{\mathbb{C}}^{n,\psi}(\Psi).$$

Then for any g of the form (7) we have

$$\Omega_{\mathbb{R}\oplus\mathbb{R}}^{n,\psi}(\Phi, g) = \gamma(g, \psi)\Omega_{\mathbb{C}}^{n,\psi}(\Psi, g).$$

For proof see §§6.2.

By substituting 0 in place of Φ or Ψ we obtain the following corollary

Corollary 6.0.4 (Density)**.** *Let $\Phi \in \mathcal{S}^{\det,\mathbb{R}^\times}(H^n(D))$. Suppose that $\Omega_D^{n,\psi}(\Phi) = 0$. Then $\Omega_D^{n,\psi}(\Phi, g) = 0$ for any relevant $g \in D$.*

For the proof of Theorem 6.0.3 we will need the following lemma, which is a more elementary version of the inversion formula.

Lemma 6.0.5. *Let $n > 1$. For any $\Phi \in \mathcal{S}(H^n(D))$, define the function f_Φ on $\mathbb{R}^\times$ by $f_\Phi(a) := \Omega_D^{n,\psi}(\Phi, aw_n)$. Then $f_\Phi \in \mathcal{S}(\mathbb{R}^\times)$ and*

$$f_\Phi(a) = |a|^{-n^2+1} \int \Omega_D^{n,\overline{\psi}}(\mathcal{F}(\Phi), \begin{pmatrix} -a^{-1}w_{n-1} & 0 \\ 0 & b \end{pmatrix})db.$$

6.1. **Proof of Lemma 6.0.5.**

Notation 6.1.1.

- *We denote $V := \{\{a_{i,j}\} \in H | a_{i,j} = 0 \text{ if } i + j \leq n + 1\} \subset H$.*
- *Note that $V^\perp = \{\{a_{i,j}\} \in H | a_{i,j} = 0 \text{ if } i + j < n + 1\} \subset H$.*
- *We denote $e := \{e_{i,j}\} \in H$. where $e_{i,j} = \delta_{i+j,n}$.*

The following two lemmas follow from change of variables.

Lemma 6.1.2. *We have*

$$f_\Phi(a) = |a|^{(n-n^2)/2} \int_{v \in V} \Phi(aw_n + v)\psi(< a^{-1}e, v >)dv$$

Lemma 6.1.3. *We have*

$$\int \Omega_D^{n,\psi}(\Phi, \begin{pmatrix} aw_{n-1} & 0 \\ 0 & b \end{pmatrix})db = |a|^{-(n+n^2)/2+1} \int_{v \in V^\perp} \Phi(ae + v)\psi(< a^{-1}w, v >)dv.$$

Lemma 6.1.4. *The function f_Φ is in $\mathcal{S}(\mathbb{R}^\times)$.*

Proof. Let $W = \mathrm{Span}(w_n) \oplus V$. Let $\Xi = \Phi|_W \in \mathcal{S}(W)$. Let $\hat{\Xi}_V \in \mathcal{S}(\mathrm{Span}(w_n) \oplus V^*)$ be the partial Fourier transform of Ξ w.r.t. V. For any $a \in \mathbb{R}^\times$ let $\phi(a) \in V^*$ be the functional defined by $\phi(a)(v) = \langle ae, v \rangle$. Consider the closed embedding $\varphi : \mathbb{R}^\times \to \mathrm{Span}(w_n) \oplus V^*$ defined by $\varphi(a) = (a, \phi(a^{-1}))$. Now by Lemma 6.1.2, $f_\Phi = \hat{\Xi}_V \circ \varphi \in \mathcal{S}(\mathbb{R}^\times)$. ☐

Proof of Lemma 6.0.5. It is left to prove that

$$f_\Phi(a) = |a|^{-n^2+1} \int \Omega_D^{n,\overline{\psi}}(\mathcal{F}(\Phi), \begin{pmatrix} -a^{-1}w_{n-1} & 0 \\ 0 & b \end{pmatrix})db.$$

Let $\delta_{ae+V} \in \mathcal{S}(H)$ and $\delta_{aw_n+V^\perp} \in \mathcal{S}(H)$ be the Haar measures on $ae + V$ and $aw_n + V^\perp$ correspondingly. Let $f_a, g_a \in C^\infty(H)$ be defined by $f_a(x) = \psi(< ae, x >)$ and $g_a(x) = \psi(< aw_n, x >)$. By Lemmas 6.1.2 and 6.1.3 the assertion follows from the fact that

$$\delta_{ae+V} g_{-a^{-1}} = \mathcal{F}^*(\delta_{-a^{-1}w_n+V^\perp} f_a).$$

☐

6.2. **Proof of Theorem 6.0.3.**

We prove the theorem by induction on n. From now on we suppose that it holds for every $r < n$.

Lemma 6.2.1. *It is enough to prove Theorem 6.0.3 for the case $\Phi \in \mathcal{S}(H^n(\mathbb{R} \oplus \mathbb{R}))$ and $\Psi \in \mathcal{S}(H^n(\mathbb{C}))$.*

Proof. Suppose that there exist $\Phi \in \mathcal{S}^{\det,\mathbb{R}^\times}(H^n(\mathbb{R} \oplus \mathbb{R}))$ and $\Psi \in \mathcal{S}^{\det,\mathbb{R}^\times}(H^n(\mathbb{C}))$ that form a counterexample for Theorem 6.0.3. We have to show that then there exist $\Phi' \in \mathcal{S}(H^n(\mathbb{R} \oplus \mathbb{R}))$ and $\Psi' \in \mathcal{S}(H^n(\mathbb{C}))$ that also form a counterexample.

We have $\Omega_{\mathbb{R}\oplus\mathbb{R}}^{n,\psi}(\Phi) = \gamma\Omega_{\mathbb{C}}^{n,\psi}(\Psi)$ but $\Omega_{\mathbb{R}\oplus\mathbb{R}}^{n,\psi}(\Phi, g) \neq \gamma(g, \psi)\Omega_{\mathbb{C}}^{n,\psi}(\Psi, g)$ for some g. Let $f \in C_c^\infty(\mathbb{R})$ such that $f(\det(g)) = 1$. Let $f' := f \circ \det$, and define $\Phi' := f'\Phi$ and $\Psi' := f'\Psi$. Note that Φ' and Ψ' are Schwartz functions and form a counterexample since determinant is invariant under the action of N. ☐

Lemma 6.2.2. *Let $\Phi \in \mathcal{S}(H^n(\mathbb{R} \oplus \mathbb{R}))$ and $\Psi \in \mathcal{S}(H^n(\mathbb{C}))$ such that $\Omega_{\mathbb{R}\oplus\mathbb{R}}^{n,\psi}(\Phi) = \gamma\Omega_{\mathbb{C}}^{n,\psi}(\Psi)$. Let $g = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, where $x \in S^i(D)$ and $y \in H^{n-i}(D)$. Then $\Omega_{\mathbb{R}\oplus\mathbb{R}}^{n,\psi}(\Phi, g) = \gamma(g, \psi)\Omega_{\mathbb{C}}^{n,\psi}(\Psi, g)$.*

This lemma follows from the induction hypotheses using intermediate Kloostermann integrals, i.e. integration over $N_i^n(D)$ (cf. §§3.2.1).

Lemma 6.2.3. *Let $\Phi \in \mathcal{S}(H^n(\mathbb{R}\oplus\mathbb{R}))$ and $\Psi \in \mathcal{S}(H^n(\mathbb{C}))$ such that $\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi) = \gamma\Omega^{n,\psi}_{\mathbb{C}}(\Psi)$. Let $g = aw_n$ where $a \in \mathbb{R}^\times$. Then $\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi, g) = \gamma(g,\psi)\Omega^{n,\psi}_{\mathbb{C}}(\Psi, g)$.*

Proof. Indeed

$$\Omega^{n,\psi}_{\mathbb{R}\oplus\mathbb{R}}(\Phi, g) = |a|^{-n^2+1}\int \Omega^{n,\overline{\psi}}_{\mathbb{R}\oplus\mathbb{R}}(\mathcal{F}(\Phi), \begin{pmatrix} -a^{-1}w_{n-1} & 0 \\ 0 & b\end{pmatrix})db =$$
$$= |a|^{-n^2+1}\int c(\psi,\mathbb{C})^{-n(n-1)/2}\gamma(\begin{pmatrix} -a^{-1}w_{n-1} & 0 \\ 0 & b\end{pmatrix}, \overline{\psi})\Omega^{n,\overline{\psi}}_{\mathbb{C}}(\mathcal{F}(\Psi), \begin{pmatrix} -a^{-1}w_{n-1} & 0 \\ 0 & b\end{pmatrix})db =$$
$$= c(\psi,\mathbb{C})^{-n(n-1)/2}\gamma(-a^{-1}w_{n-1}, \overline{\psi})\operatorname{sign}(\det(-a^{-1}w_{n-1}))\Omega^{n,\psi}_{\mathbb{C}}(\Psi, g) = \gamma(g,\psi)\Omega^{n,\psi}_{\mathbb{C}}(\Psi, g).$$

Here, the first and the third equality follow from Lemma 6.0.5, while the second equality follows from the previous lemma (Lemma 6.2.2) and the inversion formula (see Corollary 3.2.7). The last equality holds by definition of γ (Notation 6.0.1). ☐

The theorem follows now from the last 3 lemmas.

Appendix A. Schwartz functions on Nash manifolds

In this appendix we give some complementary facts about Nash manifolds and Schwartz functions on them and prove Property 2.2.7 and Theorems 2.2.15 and 2.2.13 from the preliminaries.

Theorem A.0.1 (Local triviality of Nash manifolds)**.** *Any Nash manifold can be covered by finite number of open submanifolds Nash diffeomorphic to $\mathbb{R}^n$.*

For proof see [Shi87, Theorem I.5.12].

Theorem A.0.2. *[Nash tubular neighborhood] Let M be a Nash manifold and $Z \subset M$ be closed Nash submanifold. Then there exists an finite cover $Z = \cup Z_i$ by open Nash submanifolds of Z, and open embeddings $N^M_{Z_i} \hookrightarrow M$ that are identical on the zero section.*

This follows from e.g. [AG08, Corollary 3.6.3].

Notation A.0.3. *We fix a system of semi-norms on $\mathcal{S}(\mathbb{R}^n)$ in the following way:*

$$\mathfrak{N}_k(f) := \max_{\{\alpha\in\mathbb{Z}^n_{\geq 0}\,|\,|\alpha|\leq k\}}\ \max_{\{\beta\in\mathbb{Z}^n_{\geq 0}\,|\,|\beta|\leq k\}}\ \sup_{x\in\mathbb{R}^n} |x^\alpha \frac{\partial^{|\beta|}}{(\partial x)^\beta} f|.$$

Notation A.0.4. *For any Nash vector bundle E over X we denote by $\mathcal{S}(X,E)$ the space of Schwartz sections of E.*

The properties of Schwartz functions on Nash manifolds listed in the preliminaries hold also for Schwartz sections of Nash bundles.

Remark A.0.5. *One can put the notion of push of Schwartz functions in a more invariant setting. Let $\phi : X \to Y$ be a morphism of Nash manifolds. Let E be a bundle on Y. Let E' be a bundle on X defined by $E' := \phi^*(E \otimes D_Y^{-1}) \otimes D_X$, where D_X and D_Y denote the bundles of densities on X and Y. Then we have a well defined map $\phi_* : \mathcal{S}(X,E') \to \mathcal{S}(Y,E)$.*

A.1. **Analog of Dixmier-Malliavin theorem.**

In this subsection we prove Property 2.2.7. Let us remind its formulation.

Theorem A.1.1. *Let $\phi : M \to N$ be a Nash map of Nash manifolds. Then multiplication defines an onto map $\mathcal{S}(M) \otimes \mathcal{S}(N) \twoheadrightarrow \mathcal{S}(M)$.*

First let us remind the formulation of the classical Dixmier-Malliavin theorem.

Theorem A.1.2 (see [DM78])**.** *Let a Lie group G acct continuously on a Fréchet space E. Then $C_c^\infty(G)E = E^\infty$, where E^∞ is the subspace of smooth vectors in E and $C_c^\infty(G)$ acts on E by integrating the action of G.*

Corollary A.1.3. *Let $L \subset V$ be finite dimensional linear spaces, and let L act on V by translations. Then $\mathcal{S}(L) * \mathcal{S}(V) = \mathcal{S}(V)$, where $*$ means convolution.*

Proof of Theorem A.1.1. Step 1. The case $N = \mathbb{R}^n$, $M = \mathbb{R}^{n+k}$, ϕ is the projection.
Follows from Corollary A.1.3 after applying Fourier transform.

Step 2. The case $N = \mathbb{R}^n$, $M = \mathbb{R}^k$, ϕ - general.
Identify N with the graph of ϕ in $N \times M$. The assertion follows now from the previous step using Property 2.2.5.

Step 3. The general case.
Follows from the previous step using partition of unity (Property 2.2.4) and local triviality of Nash manifolds (Theorem A.0.1). □

A.2. **Dual uncertainty principle.**

Notation A.2.1. *Let V be a finite dimensional real vector space. Let ψ be a non-trivial additive character of $\mathbb{R}$. Let μ be a Haar measure on V. Let $f \in \mathcal{S}(V)$ be a function. We denote by $\widehat{f} \in \mathcal{S}(V^*)$ the Fourier transform of f defined by μ and ψ.*

In this subsection we prove the following generalization of Theorem 2.2.13.

Theorem A.2.2. *Let V be a linear space, $L \subset V$ and $L' \subset V^*$ be subspaces. Suppose that $(L')_\perp \subsetneq L$. Then*

$$\mathcal{S}(V - L) + \mathcal{S}(\widehat{V^* - L'}) = \mathcal{S}(V).$$

The following lemma is obvious.

Lemma A.2.3. *There exists $f \in \mathcal{S}(\mathbb{R})$ such that f vanishes at 0 with all its derivatives and $\mathcal{F}(f)(0) = 1$.*

Corollary A.2.4. *Let L be a quadratic space. Let $V := L \oplus \mathbb{R}$ be enhanced with the obvious quadratic form. Let $g \in \mathcal{S}(L)$. Then there exists $f \in \mathcal{S}(V)$ such that $f \in \mathcal{S}(V - L)$ and $\mathcal{F}(f)|_L = g$.*

Corollary A.2.5. *Let L be a quadratic space. Let $V := L \oplus \mathbb{R}e$ be enhanced with the obvious quadratic form. Let $g \in \mathcal{S}(L)$. Let i be a natural number. Then there exists $f \in \mathcal{S}(V)$ such that $f \in \mathcal{S}(V - L)$, $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L = g$ and $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^j}|_L = 0$ for any $j < i$.*

Corollary A.2.6. *Let L be a quadratic space. Let $V := L \oplus \mathbb{R}e$ be enhanced with the obvious quadratic form. Let $g \in \mathcal{S}(L)$. Then for all i and ε there exists $f \in \mathcal{S}(V)$ such that $\mathfrak{N}_{i-1}(f) < \varepsilon$, $f \in \mathcal{S}(V - L)$, $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L = g$ and $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^j}|_L = 0$ for any $j < i$.*

Proof. Let $f \in \mathcal{S}(V)$ be s.t. $f \in \mathcal{S}(V - L)$, $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L = g$ and $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^j}|_L = 0$ for any $j < i$.

Let $f^t \in \mathcal{S}(V)$ defined by $f^t(x + \alpha e_1) = t^{i+2} f(x + t\alpha e_1)$. It is easy to see that $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L = g$ and $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^j}|_L = 0$ for any $j < i$. Also it is easy to see that $\lim_{t \to 0} \mathfrak{N}_{i-1}(f^t) = 0$. This implies the assertion. □

Corollary A.2.7. *Let L be a quadratic space. Let $V := L \oplus \mathbb{R}e$ be enhanced with the obvious quadratic form. Let $\{g_i\}_{i=0}^\infty \in \mathcal{S}(L)$. Then there exists $f \in \mathcal{S}(V)$ such that f vanishes on L with all its derivatives and $\frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L = g_i$.*

Proof. Define 3 sequences of functions $f_i, h_i \in S(V), g'_i \in S(L)$ recursively in the following way: $f_0 = 0$. $g'_i = g_i - \frac{\partial^i \mathcal{F}(f_{i-1})}{(\partial e)^i}|_L$. Let $h_i \in \mathcal{S}(V)$ s.t. $h_i \in \mathcal{S}(V - L)$, $\mathfrak{N}_{i-1}(h_i) < 1/2^i$, $\frac{\partial^i \mathcal{F}(h_i)}{(\partial e)^i}|_L = g'_i$ and $\frac{\partial^i \mathcal{F}(h_i)}{(\partial e)^j}|_L = 0$ for any $j < i$. Define $f_i = f_{i-1} + h_i$.

Clearly $f := \lim_{i \to \infty} f_i$ exists and satisfies the requirements. □

Corollary A.2.8. *Let L be a quadratic space. Let $V := L \oplus \mathbb{R}e$ be enhanced with the obvious quadratic form.*

Then $\mathcal{S}(V - L) + \mathcal{F}(\mathcal{S}(V - L)) = \mathcal{S}(V)$.

Proof. Let $f \in \mathcal{S}(V)$. Let $f' \in \mathcal{S}(V-L)$ s.t. $\frac{\partial^i \mathcal{F}(f')}{(\partial e)^i}|_L = \frac{\partial^i \mathcal{F}(f)}{(\partial e)^i}|_L$. Let $f'' = f - f'$. Clearly $f'' \in \mathcal{F}(\mathcal{S}(V-L))$. ☐

Corollary A.2.9. *Let V be a linear space, $L \subset V$ and $L' \subset V^*$ be subspaces of codimension 1. Suppose that $(L')_\perp \nsubseteq L$. Then*

$$\mathcal{S}(V-L) + \mathcal{S}(\widehat{V^* - L'}) = \mathcal{S}(V).$$

Proof. Choose a non-degenerate quadratic form on V s.t. $L \perp (L')_\perp$. This form gives an identification $V \to V^*$ which maps L to L'. Now the corollary follows from the previous corollary. ☐

Now we are ready to prove Theorem A.2.2.

Proof of Theorem A.2.2. Let $M \supset L$ be a sub-space in V of codimension 1 s.t. $M^\perp \nsubseteq L'$. Let $M' \supset L'$ be a sub-space in V^* of codimension 1 s.t. $M^\perp \nsubseteq M'$. The theorem follows now from the previous corollary. ☐

Appendix B. Coinvariants in Schwartz functions

Definition B.0.1. *Let a Nash group G act on a Nash manifold X. A* **tempered G-equivariant bundle** *E over X is a Nash bundle E with an equivariant structure $\phi : a^*(E) \to p^*(E)$ (here $a : G \times X \to X$ is the action map and $p : G \times X \to X$ is the projection) such that ϕ corresponds to a tempered section of the bundle* $\mathrm{Hom}(a^*(E), p^*(E))$ *(for the definition of tempered section see e.g.* [AG08]*), and for any element α in the Lie algebra of G the derivation map $a(\alpha) : C^\infty(X,E) \to C^\infty(X,E)$ preserves the sub-space of Nash sections of E.*

In this subsection we prove the following generalization of Theorem 2.2.15.

Theorem B.0.2. *Let a connected algebraic group G act on a real algebraic manifold X. Let Z be a G-invariant Zariski closed subset of X. Let $\mathfrak{g}$ be the Lie algebra of G. Let E be a tempered G-equivariant bundle over X. Suppose that for any $z \in Z$ and $k \in \mathbb{Z}_{\geq 0}$ we have*

$$(E|_z \otimes \mathrm{Sym}^k(CN^X_{Gz,z}) \otimes ((\Delta_G)|_{G_z}/\Delta_{G_z}))_{\mathfrak{g}_z} = 0.$$

Then

$$(\mathcal{S}(X,E)/\mathcal{S}(X-Z,E))_{\mathfrak{g}} = 0.$$

For the proof of this theorem we will need some auxiliary results.

Lemma B.0.3. *Let V be a representation of a Lie algebra $\mathfrak{g}$. Let F be a finite $\mathfrak{g}$-invariant filtration of V. Suppose $gr_F(V)_{\mathfrak{g}} = 0$. Then $V_{\mathfrak{g}} = 0$.*

The proof is evident by induction on the length of the filtration.

Lemma B.0.4. *Let V be a representation of a finite dimensional Lie algebra $\mathfrak{g}$. Let F_i be a countable decreasing $\mathfrak{g}$-invariant filtration of V. Suppose $\bigcap F^i(V) = 0$, $F^0(V) = V$ and that the canonical map $V \to \varprojlim (V/F^i(V))$ is an isomorphism. Suppose also that $gr^i_F(V)_{\mathfrak{g}} = 0$. Then $V_{\mathfrak{g}} = 0$.*

This lemma is standard and we included its proof for the sake of completeness.

Proof. We have to prove that the map $\mathfrak{g} \otimes V \to V$ is onto. Let $v \in V$. We will construct in an inductive way a sequence of vectors $w_i \in \mathfrak{g} \otimes V/F^i(V)$ s.t. their image under the action map $\mathfrak{g} \otimes V/F^i(V) \to V/F^i(V)$ coincides with the image of v under the quotient map $V \to V/F^i(V)$. Define $w_0 = 0$. Suppose we have already defined w_n and we have to define w_{n+1}. Let w'_{n+1} be an arbitrary lifting of w_n to $\mathfrak{g} \otimes V/F^{n+1}(V)$. Let v'_{n+1} be the image of w'_{n+1} under the action map $\mathfrak{g} \otimes V/F^{n+1}(V) \to V/F^{n+1}(V)$ and let v_{n+1} be the image of v under the quotient map $V \to V/F^{n+1}(V)$. Let $dv = v_{n+1} - v'_{n+1}$. Clearly dv lies in $F^n(V)/F^{n+1}(V)$. Let dw be its lifting to $\mathfrak{g} \otimes (F^n(V)/F^{n+1}(V))$. Denote $w_{n+1} = w'_{n+1} + dw$.

Since $\mathfrak{g}$ is finite dimensional, the canonical map $\mathfrak{g} \otimes V \to \varprojlim \mathfrak{g} \otimes (V/F^i(V))$ is an isomorphism. Therefore there exists a unique $w \in \mathfrak{g} \otimes V$ s.t. its image in $\mathfrak{g} \otimes (V/F^i(V))$ is w_i. Thus the image of w under the map $\mathfrak{g} \otimes V \to V$ is v. ☐

Notation B.0.5. *Let Z be a locally closed semi-algebraic subset of a Nash manifold X. Let E be a Nash bundle over X. Denote*

$$\mathcal{S}_X(Z,E) := \mathcal{S}(X - (\overline{Z} - Z), E)/\mathcal{S}(X - \overline{Z}, E).$$

Here we identify $\mathcal{S}(X - \overline{Z}, E)$ with a closed subspace of $\mathcal{S}(X - (\overline{Z} - Z), E)$ using the description of Schwartz functions on an open set (property 2.2.3).

Lemma B.0.6. *Let Y be a locally closed semi-algebraic subset of a Nash manifold X. Let $U \subset X$ be an open semi-algebraic subset of X that includes Y. Then extension by zero defines an isomorphism $\mathcal{S}_U(Y) \to \mathcal{S}_X(Y)$.*

Proof. Replacing X by $X - (\overline{Z} - Z)$ and U by $U - (\overline{Z} - Z)$ we may assume that Z is closed in X (and in U). Clearly the extension by zero defines an injection $e : \mathcal{S}_U(Y) \to \mathcal{S}_X(Y)$. In order to show that the extension map is onto we have to show that $\mathcal{S}(X) = \mathcal{S}(U) + \mathcal{S}(X - Y)$. This follows from partition of unity (Property 2.2.4).

□

Corollary B.0.7. *Let X be a Nash manifold and $Z \subset X$ be a locally closed semi-algebraic subset. Let E be a Nash bundle over X. Let S_i be a finite stratification of Z by locally closed semi-algebraic subsets, i.e. $Z_i := \bigcup_{j=1}^{i} S_i$ is closed in Z for every i. Then $\mathcal{S}_X(Z,E)$ has a canonical descending filtration s.t.*

$$gr_i(\mathcal{S}_X(Z,E)) \cong \mathcal{S}_X(S_i,E)).$$

Proof. Replacing X by $X - (\overline{Z} - Z)$ we may assume that Z is closed. Let $\mathcal{S}_X(Z,E)_i := \mathcal{S}(X - Z_{i-1}, E)/\mathcal{S}(X - Z, E) \subset \mathcal{S}_X(Z,E)$. Then $gr_i(\mathcal{S}_X(Z,E)) = \mathcal{S}_X(Z,E)_i/\mathcal{S}_X(Z,E)_{i+1} = \mathcal{S}(X - Z_{i-1}, E)/\mathcal{S}(X - Z_i, E)$ and $\mathcal{S}_X(Z_i,E) = \mathcal{S}(X,E)/\mathcal{S}(X - Z_i, E)$. Taking $U = X - Z_{i-1}$ and $Y = S_i$ in the previous lemma we get $\mathcal{S}(X - Z_{i-1}, E)/\mathcal{S}(X - Z_i, E) \cong \mathcal{S}(X,E)/\mathcal{S}(X - Z_i, E)$ and hence $gr_i(\mathcal{S}_X(Z,E)) \cong \mathcal{S}_X(Z_i,E))$. □

Lemma B.0.8. *Let X be a Nash manifold and $Z \subset X$ be Nash submanifold. Then $\mathcal{S}_X(Z)$ has a canonical countable decreasing filtration satisfying $\bigcap(\mathcal{S}_X(Z))^i = 0$ s.t. $gr_i(\mathcal{S}_X(Z,E)) \cong \mathcal{S}(Z, \mathrm{Sym}^i(CN_Z^X) \otimes E)$.*

Proof. It follows from the proof of Corollary 5.5.4. in [AG08].

□

Lemma B.0.9 (E. Borel)**.** *Let X be a Nash manifold and $Z \subset X$ be Nash submanifold. Then the natural map*

$$\mathcal{S}_X(Z,E) \to \lim_{\leftarrow}(\mathcal{S}_X(Z,E))/\mathcal{S}_X(Z,E))^i)$$

is an isomorpihsm.

Proof. Step 1. Reduction to the case when X is a total space of a bundle over Z.
It follows immediately from the existence of Nash tubular neighborhood (Theorem A.0.2).

Step 2. Reduction to the case when $Z = \mathbb{R}^n$ is standardly embedded inside $X = \mathbb{R}^{n+k}$.
It follows immediately from local triviality of Nash manifolds (Theorem A.0.1) and partition of unity (Property 2.2.4).

Step 3. Proof for the case when $Z = \mathbb{R}^n$ standardly embedded inside $X = \mathbb{R}^{n+k}$.
It is the same as the proof of the classical Borel Lemma. □

Definition B.0.10. *We call an action of a Nash group G on a Nash manifold X factorisable if the map $\phi_{G,X} : G \times X \to X \times X$ defined by $(g,x) \mapsto (gx,x)$ has a Nash image and is a submersion onto it.*

Theorem B.0.11 (Chevalley)**.** *Let a real algebraic group act on a real algebraic variety X. Then there exists a finite G-invariant smooth stratification X_i of X s.t. the action of G on X_i is factorizable.*

Proof. By the classical Chevalley Theorem there exists a Zariski open subset $U \subset G \times X$ s.t. the map $\phi_{G,X}|_U$ is a submersion to its smooth image. Let $X_0 \subset X$ be the projection of U to X. It is easy to see that $\phi_{G,X}|_{G \times X_0}$ is a submersion to its smooth image. The theorem now follows by Noetherian induction. □

Theorem B.0.12. *Let a Nash group G act factorizably on a Nash manifold X and E be a tempered G-equivariant bundle over X. Suppose that for any $x \in X$ we have*

$$((E|_x \otimes ((\Delta_G)|_{G_x}/\Delta_{G_x})))_{\mathfrak{g}_x} = 0.$$

Then

$$(\mathcal{S}(X,E))_{\mathfrak{g}} = 0.$$

For the proof see section B.1 below.

Now we ready to prove Theorem B.0.2.

Proof of Theorem B.0.2.
Step 1. Reduction to the case when the action of G on Z is factorizable.
By Theorem B.0.11, in the general case there is a stratification $Z = \bigcup_{i=1}^n S_i$ such that the action on each strata S_i is factorizable. By Corollary B.0.7, the associated graded parts of this filtration are isomorphic to $\mathcal{S}_X(Z_i, E)$. The reduction follows now from Lemma B.0.3.

Step 2. Reduction to the case when the action of G on Z is factorizable and $Z = X$.
It follows from the Borel Lemma (Lemma B.0.9) and Lemma B.0.4.

Step 3. Proof for the case that the action of G on Z is factorizable and $Z = X$.
It follows from Theorem B.0.12. □

B.1. **Proof of Theorem B.0.12.**

B.1.1. *A sketch of the proof.* The proof is rather technical, so let us start with a brief description of the main steps of the proof. First we give a geometric description of the space of co-invariants of an action of a Nash group G on the space of Schwartz sections of a G-equivariant bundle, see Corollary B.1.9 below. In the notation of Theorem B.0.12 this corollary describes $\mathfrak{g}\mathcal{S}(X,E)$.

Then we try to generalize this description to the case of Nash family of groups (see Definition B.1.10 below) and furthermore to the case of Nash family of Nash torsors, i.e. a family of spaces which will become groups after a choice of a point in each space (see Definition B.1.21 below). Unfortunately we can not generalize Corollary B.1.9 completely for this case, but we can easily obtain its partial analog (see Corollary B.1.17 below).

Now we attach to our situation a family of torsors parameterized by an appropriate subset of $X \times X$. Namely for each pair of points in X we consider the subset of G consisting of elements which connect those points.

Then we use our descriptions of the spaces of co-invariants and conclude that in order to show the vanishing of $\mathfrak{g}$ co-invariants of $\mathcal{S}(X,E)$ it is enough to show vanishing of co-invariants of a certain family of finite dimensional representations of the family of torsors described above.

This follows from the conditions of the theorem using Lemma B.1.18 below.

B.1.2. *A description of the space of co-invariants.*

Notation B.1.1. *Let $\phi : X \to Y$ be a map of (Nash) manifolds.*
(i) Denote $D_Y^X := D_\phi := \phi^*(D_Y^*) \otimes D_X$.
(ii) Let $E \to Y$ be a (Nash) bundle. Denote $\phi^?(E) = \phi^*(E) \otimes D_Y^X$.

Remark B.1.2. *Note that*
(i) If ϕ is a submersion then for all $y \in Y$ we have $D_Y^X|_{\phi^{-1}(y)} \cong D_{\phi^{-1}(y)}$.
(ii) If ϕ is a submersion then by Remark A.0.5 we have a well defined map $\phi_* : \mathcal{S}^*(X, \phi^?(E)) \to \mathcal{S}^*(Y,E)$.
(iii) If a Lie group G acts on a smooth manifold X and E is a G-equivariant vector bundle (i.e. we have a map $p^*(E) \to a^*(E)$, where $p : G \times X \to X$ is the projection and $a : G \times X \to X$ is the action) then we also have a natural map $p^?(E) \to a^?(E)$. If G, X and E are Nash and the actions of G on X and E are Nash then the map $p^?(E) \to a^?(E)$ is Nash. If the action of G on E is tempered then the map $p^?(E) \to a^?(E)$ corresponds to a tempered section of $\operatorname{Hom}(p^?(E), a^?(E))$.

Notation B.1.3. *Let G be a Nash group. We denote*

$$\mathcal{S}(G, D_G)_0 := \{f \in \mathcal{S}(G, D_G) | \int_G f = 0\}.$$

Lemma B.1.4. *Let G be a connected Nash group and $\mathfrak{g}$ be its Lie algebra. Then $\mathfrak{g}\mathcal{S}(G, D_G) = \mathcal{S}(G, D_G)_0$.*

For the proof we will need the following lemma which is a special case of [AG10, Theorem 3.1.8].

Lemma B.1.5. *Let X be an n-dimensional connected orientable Nash manifold. Let $0 \leq i \leq n$ and let Ω^i_X denote the bundle of differential i-forms on X. Then the image of the De-Rham differential $d : \mathcal{S}(X, \Omega^{n-1}_X) \to \mathcal{S}(X, \Omega^n_X)$ is of co-dimension 1.*

Proof of lemma B.1.4. The inclusion $\mathfrak{g}\mathcal{S}(G, D_G) \subset \mathcal{S}(G, D_G)_0$ is evident. It remains to prove that $\mathfrak{g}\mathcal{S}(G, D_G)$ is of co-dimension 1 in $\mathcal{S}(G, D_G)$.

The proof of this fact is analogous to the proof of Proposition 4.0.11 in [AG10]. For the sake of completeness, let us perform it. Since G is orientable we will identify Ω^n_G with D_G. By lemma B.1.5 it is enough to prove that the image of the derivation map $\phi : \mathfrak{g} \otimes \mathcal{S}(G, D_G) \to \mathcal{S}(G, D_G)$ coincides with the image of $d : \mathcal{S}(X, \Omega^{n-1}_X) \to \mathcal{S}(X, \Omega^n_X)$. Note the the map $\psi : \mathfrak{g} \otimes \mathcal{S}(G, D_G) \to \mathcal{S}(X, \Omega^{n-1}_X)$ defined by $\psi(\alpha, \omega) = i_\alpha(\omega)$ is surjective. By the homotopy formula for Lie derivative, for any $\alpha \in \mathfrak{g}$ and any $\omega \in \mathcal{S}(G, \Omega^n_G)$ we have $\alpha\omega = d(i_\alpha(\omega))$. Therefore $\phi = d \circ \psi$. This proves that $Im\phi = Imd$. ☐

Notation B.1.6. *Let G be a Nash group, X be a Nash manifold and E be a Nash bundle over X. Let $p : G \times X \to X$ be the projection. Denote by $\mathcal{S}(G \times X, p^?(E))_{0,X}$ the kernel of the map*

$$p_* : \mathcal{S}(G \times X, p^?(E)) \to \mathcal{S}(X, E).$$

In cases when there is no ambiguity we will denote it just by $\mathcal{S}(G \times X, p^?(E))_0$.

Lemma B.1.7. *Let G be a Nash group, X be a Nash manifold and E be a Nash bundle over X. Let $p : G \times X \to X$ be the projection. Then*

$$\mathcal{S}(G \times X, p^?(E))_{0,X} \cong \mathcal{S}(G, D_G)_0 \widehat{\otimes} \mathcal{S}(X, E).$$

Proof. The sequence

$$0 \to \mathcal{S}(G, D_G)_0 \to \mathcal{S}(G, D_G) \to \mathbb{C} \to 0$$

is exact. Therefore by Proposition 2.3.2 the sequence

$$0 \to \mathcal{S}(G, D_G)_0 \widehat{\otimes} \mathcal{S}(X, E) \to \mathcal{S}(G, D_G) \widehat{\otimes} \mathcal{S}(X, E) \to \mathcal{S}(X, E) \to 0$$

is also exact. Thus it is enough to show that the map $\mathcal{S}(G, D_G) \widehat{\otimes} \mathcal{S}(X, E) \to \mathcal{S}(X, E)$ corresponds to the map $p_* : \mathcal{S}(G \times X, p^?(E)) \to \mathcal{S}(X, E)$ under the identification $\mathcal{S}(G, D_G) \widehat{\otimes} \mathcal{S}(X, E) \cong \mathcal{S}(G \times X, p^?(E))$. Since those maps are continuous it is enough to check that they are the same on the image of $\mathcal{S}(G, D_G) \otimes \mathcal{S}(X, E)$, which is evident. ☐

Corollary B.1.8. *Let G be a Nash group, X be a Nash manifold and E be a Nash bundle over X. Let $\mathfrak{g}$ be the Lie algebra of G. Let $p : G \times X \to X$ be the projection. Let G act on $\mathcal{S}(G \times X, p^?(E))$ by acting on the G coordinate. Then $\mathfrak{g}\mathcal{S}(G \times X, p^?(E)) = \mathcal{S}(G \times X, p^?(E))_{0,X}$.*

Proof. By Lemma B.1.4, the derivation map $\phi : \mathfrak{g} \otimes \mathcal{S}(G, D_G) \to \mathcal{S}(G, D_G)_0$ is surjective. Thus by Proposition 2.3.2 so is the derivation map $\phi' : \mathfrak{g} \otimes \mathcal{S}(G, D_G) \widehat{\otimes} \mathcal{S}(X, E) \to \mathcal{S}(G, D_G)_0 \widehat{\otimes} \mathcal{S}(X, E)$. Therefore by the last lemma the derivation map $\phi'' : \mathfrak{g} \otimes \mathcal{S}(G \times X, p^?(E)) \to \mathcal{S}(G \times X, p^?(E))_{0,X}$ is surjective too. ☐

Corollary B.1.9. *Let G be a connected Nash group and $\mathfrak{g}$ be its Lie algebra. Let G act on a Nash manifold X and let E be a tempered G-equivariant bundle over X. Let $p : G \times X \to X$ be the projection. Let $a : G \times X \to X$ be the action map.*

Then $\mathfrak{g}\mathcal{S}(X, E)$ is the image $a_*(\mathcal{S}(G \times X, a^?(E))_{0,X,a})$ where $\mathcal{S}(G \times X, a^?(E))_{0,X,a}$ denotes the image of $\mathcal{S}(G \times X, p^?(E))_{0,X}$ under the identification $\mathcal{S}(G \times X, p^?(E)) \cong \mathcal{S}(G \times X, a^?(E))$.

Proof. Let G act on $\mathcal{S}(G \times X, p^?(E))$ by acting on the G coordinate. The identification $\mathcal{S}(G \times X, p^?(E)) \cong \mathcal{S}(G \times X, a^?(E))$ gives us an action of G on $\mathcal{S}(G \times X, a^?(E))$. It is easy to see that $a_* : \mathcal{S}(G \times X, a^?(E)) \to \mathcal{S}(X, E)$ is a morphism of G-representations. By property 2.2.6 a_* is surjective. Therefore $(\mathfrak{g}\mathcal{S}(X, E)) = a_*((\mathfrak{g}\mathcal{S}(G \times X, a^?(E)))$. The assertion follows now by the previous corollary. ☐

B.1.3. *Nash family of groups.*

Definition B.1.10. *A* ***Nash family of groups*** *over a Nash manifold X is a surjective submersion $G \to X$, a Nash map $m : G \times_X G \to G$ and a Nash section $e : X \to G$ s.t. for any $x \in X$ the map $m|_{G|_x \times G|_x}$ gives a group structure on the fiber $G|_x$ and $e(x)$ is the unit of this group.*

Definition B.1.11. *A* ***Nash family of Lie algebras*** *over a Nash manifold X is a Nash bundle $\mathfrak{g} \to X$, a Nash section m of the bundle $\mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ s.t. for any $x \in X$ the map $m(x) : \mathfrak{g}|_x \otimes \mathfrak{g}|_x \to \mathfrak{g}|_x$ gives a Lie algebra structure on the fiber $\mathfrak{g}|_x$.*

Definition B.1.12. *A Nash family of Lie algebras of a Nash family of groups G over a Nash Manifold X is the bundle $e^*(N^G_{e(X)})$ equipped with the natural structure of a Nash family of Lie algebras. We will denote it by* $\mathrm{Lie}(G)$.

Notation B.1.13. *Let G be a Nash family of groups over a Nash manifold X. Let E be a bundle over X. Let $p : G \to X$ be the projection. Denote by $\mathcal{S}(G, p^?(E))_{0,X}$ the kernel of the map $p_* : \mathcal{S}(G, p^?(E))) \to \mathcal{S}(X, E)$. If there is no ambiguity we will denote it by $\mathcal{S}(G, p^?(E))_0$.*

Lemma B.1.14. *Let G be a Nash family of groups over a Nash manifold X and $\mathfrak{g}$ be its family of Lie algebras. Then the image of the natural map $\mathcal{S}(X, \mathfrak{g}) \otimes \mathcal{S}(G, D_G) \to \mathcal{S}(G, D_G)$ is included in $\mathcal{S}(G, D_G)_{0,X}$.*

Proof. It follows immediately from the case when X is one point and E is $\mathbb{C}$ which follows from Lemma B.1.4. □

Definition B.1.15. *A Nash family of representations of a Nash family of Lie algebras $\mathfrak{g}$ over a Nash manifold X is a bundle E over X and a Nash section a of the bundle $\mathrm{Hom}(\mathfrak{g} \otimes E, E)$ s.t. for any $x \in X$ the map $a(x) : \mathfrak{g}|_x \otimes E|_x \to E|_x$ gives a structure of a representation of $\mathfrak{g}|_x$ on the fiber $E|_x$.*

Definition B.1.16. *Let G be a Nash family of groups over a Nash manifold X. Let $\mathfrak{g}$ be its family of Lie algebras. Let $p : G \to X$ be the projection. A tempered (finite dimensional) family of representations of G is a pair (E, a) where E is a Nash bundle over X and a is a tempered section of the bundle $\mathrm{End}(p^*E)$ s.t. for any $x \in X$ the section $a|_{G|_x}$ gives a structure of a representation of $G|_x$ on the fiber $E|_x$ and s.t. the differential of a considered as a section of $\mathrm{Hom}(\mathfrak{g} \otimes E, E)$ gives a structure of a Nash family of representations of $\mathfrak{g}$ on E.*

Lemma B.1.14 gives us the following corollary.

Corollary B.1.17. *Let G be a Nash family of groups over a Nash manifold X and $\mathfrak{g}$ be its Lie algebra. Let (E, a) be a tempered (finite dimensional) family of representations of G. Let ϕ denote the composition $\mathcal{S}(G, p^?(E)) \overset{a}{\to} \mathcal{S}(G, p^?(E)) \overset{p_*}{\to} \mathcal{S}(X, E)$. Then the image of the natural map $\mathcal{S}(X, \mathfrak{g}) \otimes \mathcal{S}(X, E) \to \mathcal{S}(X, E)$ is included in $\phi(\mathcal{S}(G, p^?(E) \otimes D_G)_{0,X})$.*

Proof. Let V be the space $\mathcal{S}(G, p^?(E))$ equipped with the action action of the Lie algebra $\mathcal{S}(X, \mathfrak{g})$ which is given by the action a. Let W be the same space $\mathcal{S}(G, p^?(E))$ equipped with the action of the Lie algebra $\mathcal{S}(X, \mathfrak{g})$ which is given by the trivial action of G on E. Note that the maps $a : W \to V$ and $p_* : V \to \mathcal{S}(X, E)$ are $\mathcal{S}(X, \mathfrak{g})$-equivariant and surjective. The assertion follows now from Lemma B.1.14. □

Lemma B.1.18. *Let $\mathfrak{g}$ be a Nash family of Lie algebras over a Nash manifold X. Let E be a Nash family of its representations. Consider $\mathcal{S}(X, \mathfrak{g})$ as a Lie algebra and $\mathcal{S}(X, E)$ as its representation. Suppose that for any $x \in X$ we have $(E|_x)_{\mathfrak{g}|_x} = 0$. Then $(\mathcal{S}(X, E))_{\mathcal{S}(X,\mathfrak{g})} = 0$.*

Proof. For any $x \in X$ denote by a_x the map $\mathfrak{g}|_x \otimes E|_x \to E|_x$. By partition of unity (property 2.2.4) we may assume that E and $\mathfrak{g}$ are trivial bundles with fibers V and W. Fix a basis for V and W and the corresponding basis for $W \otimes V$. Let $\mathfrak{S}$ be the collection of coordinate subspaces of $W \otimes V$ of dimension $\dim V$. For any $L \in \mathfrak{S}$ denote $U_L = \{x \in X | a_x(L) = V\}$. Clearly $X = \bigcup U_L$. Thus by partition of unity we may assume that $X = U_L$ for some L. For this case the lemma is evident. □

Corollary B.1.19. *Let G be a Nash family of groups over a Nash manifold X and $\mathfrak{g}$ be its Lie algebra. Let (E, a) be a tempered (finite dimensional) family of representations of G. Let ϕ denote the composition*

$$\mathcal{S}(G, p^?(E)) \overset{a}{\to} \mathcal{S}(G, p^?(E)) \overset{p_*}{\to} \mathcal{S}(X, E).$$

Suppose that for any $x \in X$ we have $(E|_x)_{\mathfrak{g}_x} = 0$. Then

$$\phi(\mathcal{S}(G, p^?(E))_{0,X}) = \mathcal{S}(X, E).$$

B.1.4. *Nash family of torsors.*

Definition B.1.20. *We call a set G equipped with a map $m : G \times G \times G \to G$* ***a torsor*** *if there exists a group structure on G s.t. $m(x, y, z) = z((z^{-1}x)(z^{-1}y))$. One may say that a torsor is a group without choice of identity element.*

Definition B.1.21. *A Nash family of torsors over a Nash manifold X is a surjective submersion $G \to X$ and a Nash map $m : G \times_X G \times_X G \to G$ s.t. for any $x \in X$ the map $m|_{G|_x \times G|_x \times G|_x}$ gives a torsor structure on the fiber $G|_x$.*

Definition B.1.22. *Let G be a Nash family of torsors over a Nash manifold X. Let $p : G \to X$ be the projection. Consider* $\operatorname{Ker} dp$ *as a subbundle of TG. It has a natural structure of a family of Lie algebras over G. We will call this family the family of Lie algebras of G.*

Remark B.1.23. *One could define the family of Lie algebras of G to be a family of Lie algebras over X. This definition would be more adequate, but it is technically harder to phrase it. We did not do it since it is unnecessary for our purposes.*

Definition B.1.24. *A* **representation of a torsor** *G is a pair (V, W) of vector spaces and a morphism of torsors $G \to \operatorname{Iso}(V, W)$.*

Definition B.1.25. *Let G be a Nash family of torsors over a Nash manifold X. Let $\mathfrak{g}$ be its family of Lie algebras. Let $p : G \to X$ by the projection. A* **tempered (finite dimensional) family of representations of** *G is a triple (E, L, a), where E and L are (Nash) bundles over X and a is a tempered section of the bundle $Hom(p^*E, p^*L)$ s.t. for any $x \in X$ the section $a|_{G|_x}$ gives a structure of a representation of $G|_x$ on the fibers $E|_x$ and $L|_x$ and s.t. the differential of a considered as a section of* $\operatorname{Hom}(\mathfrak{g} \otimes p^*L, p^*L)$ *gives a structure of a Nash family of representations of $\mathfrak{g}$ on p^*L.*

Notation B.1.26. *Let G be a Nash family of torsors over a Nash manifold X. Let E be a bundle over X. Let $p : G \to X$ be the projection. Again we denote by $\mathcal{S}(G, p^?(E))_{0,X}$ the kernel of the map $p_* : \mathcal{S}(G, p^?(E))) \to \mathcal{S}(X, E)$. If there is no ambiguity we will denote it by $\mathcal{S}(G, p^?(E))_0$.*

Corollary B.1.19 gives us the following corollary.

Corollary B.1.27. *Let G be a Nash family of torsors over a Nash manifold X and $\mathfrak{g}$ be its family of Lie algebras. Let (E, L, a) be a tempered (finite dimensional) family of representations of G. Let ϕ denote the composition*

$$\mathcal{S}(G, p^?(E)) \xrightarrow{a} \mathcal{S}(G, p^?(L)) \xrightarrow{p_*} \mathcal{S}(X, L).$$

Suppose that for any $x \in G$ we have $(L|_{p(x)})_{\mathfrak{g}_x} = 0$. Then

$$\phi(\mathcal{S}(G, p^?(E))_0) = \mathcal{S}(X, L).$$

For the proof we will need [AG10, Theorem 2.4.3]. Let us recall it:

Theorem B.1.28. *Let M and N be Nash manifolds and $\nu : M \to N$ be a surjective submersive Nash map. Then locally (in the restricted topology) it has a Nash section, i.e. there exists a finite open cover $N = \bigcup_{i=1}^{k} U_i$ such that ν has a Nash section on each U_i.*

Now we ready to deduce the corollary.

Proof of Corollary B.1.27. By Theorem B.1.28 we can cover X by finitely many Nash open sets $X = \bigcup_{i=1}^{k} U_i$ such that $p|_{p^{-1}(U_i)}$ will have a section. Using partition of unity (property 2.2.4) we may assume that p have a section. thus we G becomes a Nash family of groups. The assertion follows now from Corollary B.1.19. $\square$

B.1.5. *Proof of Theorem B.0.12.* By Lemma B.1.9 it is enough to show that

$$a_*(\mathcal{S}(G \times X, a^?(E))_{0,X,a}) = \mathcal{S}(X, E).$$

Let Y be the image of the map $b : G \times X \to X \times X$ defined by $b(g, x) = (x, gx)$. Let $E_i = p_i^?(E)$ for $i = 1, 2$. Here $p_i : Y \to X$ is the projection to the i's coordinate. Note that $G \times X$ has a natural structure of a family of torsors over Y and the G-equivariant structure on E gives a family of representations (ψ, E_1, E_2) of the family of torsors $b : G \times X \to Y$. It is enough to show that

$$b_*(\mathcal{S}(G \times X, a^?(E)))_0) = \mathcal{S}(Y, E_2).$$

Recall that $a_*(\mathcal{S}(G \times X, a^?(E))_{0,X,a})$ is the image of $\mathcal{S}(G \times X, p^?(E))_{0,X}$ under the identification $\phi : \mathcal{S}(G \times X, a^?(E)) \to \mathcal{S}(G \times X, p^?(E)))$. Note that $\mathcal{S}(G \times X, p^?(E))_{0,X}$ includes $\mathcal{S}(G \times X, p^?(E))_{0,Y}$. Therefore it is enough to show that the image of $\mathcal{S}(G \times X, b^?(E_1))_{0,Y}$ under the composition

$$\mathcal{S}(G \times X, b^?(E_1)) \to \mathcal{S}(G \times X, b^?(E_2)) \to \mathcal{S}(Y, E_2)$$

is $\mathcal{S}(Y, E_2)$. This follows by Corollary B.1.27 from the fact that for every $y \in Y$ we have $((E_2)|_y)_{\mathfrak{g}_{p_2(y)}} = 0$. This fact is a reformulation of the fact that

$$((E|_{p_2(y)} \otimes ((\Delta_G)|_{G_{p_2(y)}} / \Delta_{G_{p_2(y)}})))_{\mathfrak{g}_{p_2(y)}} = 0,$$

which is part of the assumptions of the theorem. □

References

[AG08] A. Aizenbud, D. Gourevitch, *Schwartz functions on Nash Manifolds,* International Mathematics Research Notices, Vol. 2008, n.5, Article ID rnm155, 37 pages. DOI: 10.1093/imrn/rnm155. See also arXiv:0704.2891 [math.AG].

[AG10] Aizenbud, A.; Gourevitch, D.: *De-Rham theorem and Shapiro lemma for Schwartz functions on Nash manifolds,* Israel Journal of Mathematics **177** (2010), pp 155-188. See also arXiv:0802.3305[math.AG].

[AG09] Aizenbud, A.; Gourevitch, D.:*Generalized Harish-Chandra descent, Gelfand pairs and an Archimedean analog of Jacquet-Rallis' Theorem,* Duke Mathematical Journal, Volume 149, Number 3,509-567 (2009). See also arXiv: 0812.5063[math.RT].

[BCR98] Bochnak, J.; Coste, M.; Roy, M-F.: *Real Algebraic Geometry*, Berlin: Springer, 1998.

[CHM00] W. Casselman; H. Hecht; D. Miličić, *Bruhat filtrations and Whittaker vectors for real groups*, The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), 151-190, Proc. Sympos. Pure Math., **68**, Amer. Math. Soc., Providence, RI, (2000).

[FLO] Brooke Feigon, Erez Lapid and Omer Offen: *On representations distinguished by unitary groups,* preprint.

[DM78] Jacques Dixmier and Paul Malliavin, *Factorisations de fonctions et de vecteurs indefiniment differentiables.* Bull. Sci. Math. (2) **102** (1978), no. 4, 307-330. MR MR517765

[Jac98] H. Jacquet *A theorem of density for Kloosterman integrals.* Asian J. Math. 2 (1998), no. 4, 759778.

[Jac03a] H. Jacquet *Smooth transfer of Kloosterman integrals*, Duke Math. J., **120**, pp. 121-152 (2003).

[Jac03b] H. Jacquet *Facteurs de transfert pour les integrales de Kloosterman,* C. R. Math. Acad. Sci. Paris **336**, no. 2, pp. 121–124 (2003).

[Jac04] *Kloosterman identities over a quadratic extension*, Ann. of Math. (2) 160 (2004), no. 2, 755779.

[Jac05a] H. Jacquet *Kloosterman Integrals for $GL(2, R)$*, Pure and Applied Mathematics Quarterly Volume 1, Number 2, pp. 257-289, (2005).

[Jac05b] H. Jacquet, *Kloosterman identities over a quadratic extension II*, Ann. Scient. Ec. Norm. Sup., Vol. 38, Issue 4, pp. 609-669, (2005)

[JY90] H. Jacquet, Y.Ye, *Une remarque sur le changement de base quadratique,* C. R. Math. Acad. Sci. Paris **311**, no. 11, pp 671–676 (1990).

[KV96] J. A.C. Kolk and V.S. Varadarajan, *On the transverse symbol of vectorial distributions and some applications to harmonic analysis*, Indag. Mathem., N.S., 7 (l), pp 67-96 (1996).

[Ngo97] B.C. Ngo *Le lemme fondamental de Jacquet et Ye en caracteristiques egales,* C. R. Acad. Sci. Paris, **325**, Serie I, pp. 307-312, (1997).

[Ngo99] B.C. Ngo *le lemme fondamental de Jacquet et Ye en caracteristique positive*, Duke Mathematical Journal **96**, no . 3 (1999).

[Shi87] M. Shiota, *Nash Manifolds*, Lecture Notes in Mathematics **1269** (1987).

[Spr85] T. A. Springer, *Some results on algebraic groups with involutions* in Algebraic Groups and Related Topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math. 6, North-Holland, Amsterdam, , 525–543 (1985).

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