Bounds on multiplicities of spherical spaces over finite fields

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Main conjecture

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Let G be a reductive algebraic group scheme and X be a spherical G space (i.e. over any geometric point of $\mathrm{spec}(\mathbb{Z})$*, the Borel acts with finitely may orbits on X).*

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previews results

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- The multiplicities are of geometric nature and $\limsup_{n \to \infty} \dim \mathrm{Hom}(\rho, \mathbb{C}[X(\mathbb{F}_{p^n})])$ is bounded.
- Deduce the result.

Main tool – Lusztig's character sheaves

Theorem (Lusztig, Shoji)

Let G be an algebraic group of type GL defined over $\mathbb{F}_q$. For every irreducible representation ρ of $G(\mathbb{F}_q)$, there is an induced character sheaf $\mathcal{M}$ together with a Weil structure $\alpha : \mathrm{Frob}_q^* \mathcal{M} \to \mathcal{M}$ which is pure of weight zero, such that $\chi_{M,\alpha} = \chi_\rho$.

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$\mathcal{M}$ is a (perversed) direct summand of $\pi_*(\mathcal{K})$, for some line bundle $\mathcal{K}$ on $\tilde{G}$.

Dimension of the orbit space

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Let algebraic group H act on a variety Y. Denote
$Y_H := \{(y,h) \in Y \times H | hy = y\}$.

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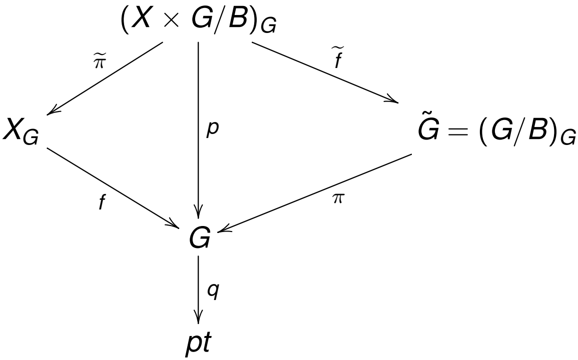
- $\mathcal{B}_G = \tilde{G}$.
- Y has finitely many orbits iff $\dim Y_H = \dim H$.
- $\dim(X \times \mathcal{B})_G = \dim G$ iff X is spherical.

Categorification of the computation of multiplicity of principal series representations



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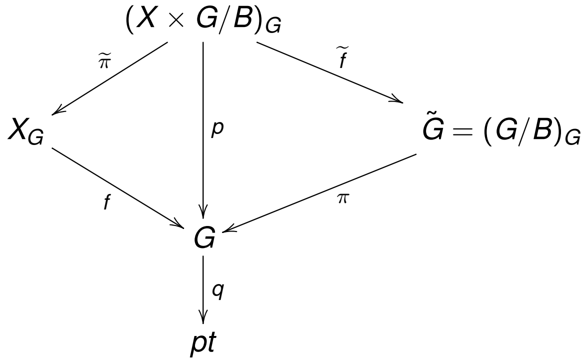

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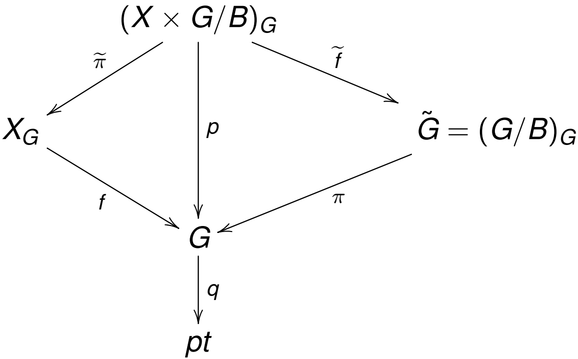

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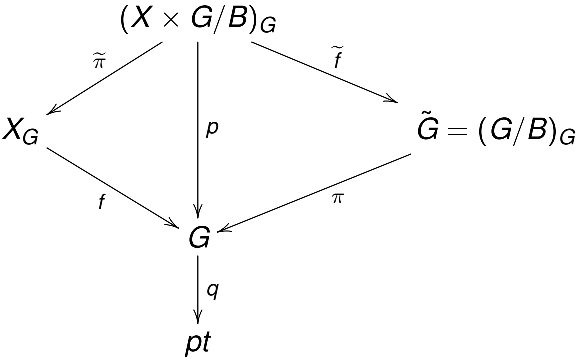

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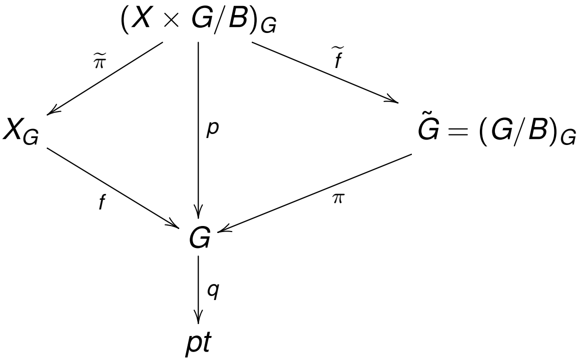

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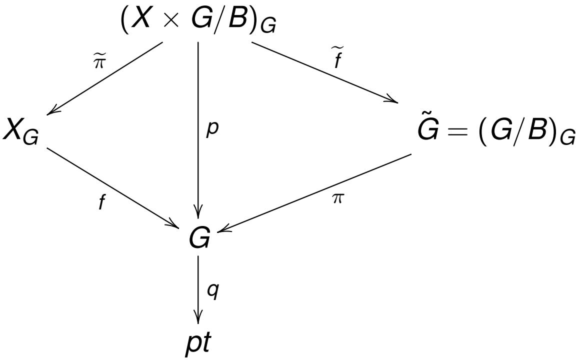

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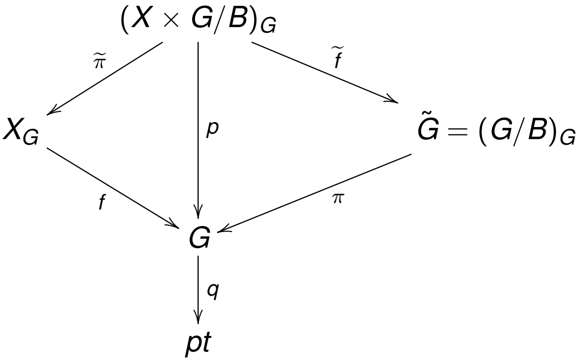

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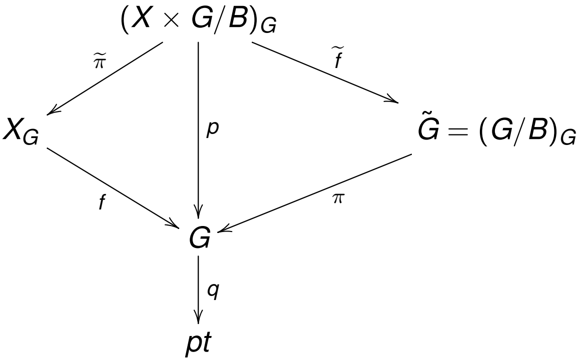

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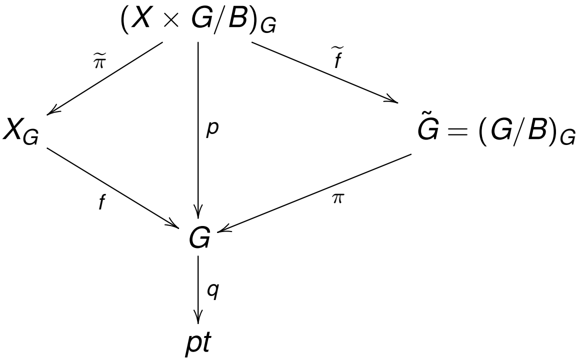

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The proof for fixed characteristic

Conclusion

We constructed a variety $Z := (X \times \mathcal{B})_G$ *of dimension* $\dim G$ *such that for any irreducible representation* $\rho \in \mathrm{irr}(G(\mathbb{F}_q))$*, there exist a representation* $\rho' \supset \rho$*, a line bundle* $\mathcal{F}$ *on* Z *and wight* ≤ 0 *Weil structure* β *on* $H^*(Z, \mathcal{F})$ *s.t.*

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- $M(n) = Q(v^n)$, where Q is a rational function on $\mathbb{C}^d$ and $v \in (\mathbb{C}^\times)^d$.

Lemma

Suppose Q is a rational function on $\mathbb{C}^d$. Let $v \in (\mathbb{C}^\times)^d$ such that Q is regular at v^n, for all $n \in \mathbb{Z}_{>0}$, and the set $\{Q(v^n) | n \in \mathbb{Z}_{>0}\}$ is finite.

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