Representation theory and the Group algebra

A. Aizenbud

Weizmann Institute of Science

http://www.wisdom.weizmann.ac.il/~aizenr

The group algebra

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Theorem

G-representations $\Longleftrightarrow$ $\mathbb{C}[G]$-modules

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  $m(A)(g) = \mathrm{Tr}(\pi(g^{-1})A)/\dim V$

□

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$$\#irr(G) = \#G//G$$

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Frobenius Formula

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Theorem (Frobenius 1896)

$$\zeta_G(2n-2) = \frac{\#\{(g_1, h_1, \dots g_n, h_n) \in G^{2n} | [g_1, h_1] \cdots [g_n, h_n] = 1\}}{\# G^{2n-1}}$$