Generalized Harish-Chandra descent

A. Aizenbud

Massachusetts Institute of Technology

http://math.mit.edu/~aizenr

Aim

Task

Let a reductive group G act on an affine variety X and let χ be a character of G.

Aim

Task

Let a reductive group G act on an affine variety X and let χ be a character of G. We want to prove:

$$\mathcal{S}^*(X)^{G,\chi} = 0.$$

Aim

Task

Let a reductive group G act on an affine variety X and let χ be a character of G. We want to prove:

$$\mathcal{S}^*(X)^{G,\chi} = 0.$$

Applications:

Aim

Task

Let a reductive group G act on an affine variety X and let χ be a character of G. We want to prove:

$$\mathcal{S}^*(X)^{G,\chi} = 0.$$

Applications: Representation theory, Harmonic analysis,

Aim

Task

Let a reductive group G act on an affine variety X and let χ be a character of G. We want to prove:

$$\mathcal{S}^*(X)^{G,\chi} = 0.$$

Applications: Representation theory, Harmonic analysis, Gelfand pairs,

Aim

Task

Let a reductive group G act on an affine variety X and let χ be a character of G. We want to prove:

$$\mathcal{S}^*(X)^{G,\chi} = 0.$$

Applications: Representation theory, Harmonic analysis, Gelfand pairs, trace formula, relative trace formula, ...

Aim

Task

Let a reductive group G act on an affine variety X and let χ be a character of G. We want to prove:

$$\mathcal{S}^*(X)^{G,\chi} = 0.$$

Applications: Representation theory, Harmonic analysis, Gelfand pairs, trace formula, relative trace formula, ...

Necessary condition:

Close orbits do not carry equivariant distributions

Aim

Task

Let a reductive group G act on an affine variety X and let χ be a character of G. We want to prove:

$$\mathcal{S}^*(X)^{G,\chi} = 0.$$

Applications: Representation theory, Harmonic analysis, Gelfand pairs, trace formula, relative trace formula, ...

Necessary condition:

Close orbits do not carry equivariant distributions

$$\Updownarrow$$

$\chi|_{G_a} \neq 1$ for any semi simple $a \in X$ (i.e. $a \in X$ with closed orbit Ga)

Distributions

Distributions

Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M. We will consider the space $(C_c^\infty(M))^*$ of distributions on M. Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M.

Distributions

Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M. We will consider the space $(C_c^\infty(M))^*$ of distributions on M. Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M.

Definition

An ℓ-space is a Hausdorff locally compact totally disconnected topological space. For an ℓ-space X we denote by $\mathcal{S}(X)$ the space of compactly supported locally constant functions on X. We let $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ be the space of distributions on X.

Frobenius descent

$$\begin{array}{ccc} X_z & \longrightarrow & X \\ \downarrow & & \downarrow \\ z & \longrightarrow & Z \end{array}$$

Frobenius descent

$$\begin{array}{ccc} X_z & \longrightarrow & X \\ \downarrow & & \downarrow \\ z & \longrightarrow & Z \end{array}$$

Theorem (Bernstein, Baruch, ...)

Let $\psi : X \to Z$ be a map.
Let G act on X and Z such that $\psi(gx) = g\psi(x)$.
Suppose that the action of G on Z is transitive.
Suppose that both G and $Stab_G(z)$ are unimodular. Then

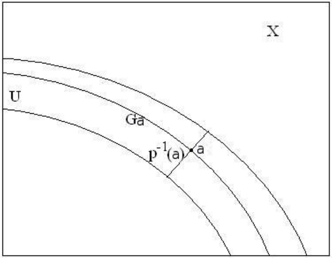
$$\mathcal{S}^*(X)^{G,\chi} \cong \mathcal{S}^*(X_z)^{Stab_G(z),\chi}.$$

Luna's slice theorem

Luna's slice theorem

Theorem (Luna)

Let a reductive group G act on a smooth affine algebraic variety X. Let $a \in X$ be a semi-simple point. Then there exist an invariant (etale) neighborhood U of Ga with an equvariant projection $p : U \to Ga$ s.t. the fiber $p^{-1}(a)$ is G-isomorphic to an (etale) neighborhood of 0 in the normal space $N^X_{Ga,a}$



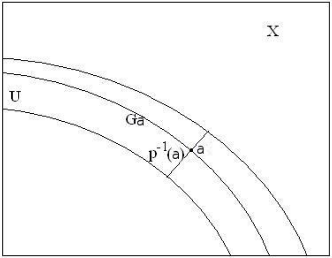
Generalized Harish-Chandra descent

Theorem (A.-Gourevitch)

Let a reductive group G act on a smooth affine algebraic variety X. Let χ be a character of G. Suppose that for any $a \in X$ s.t. the orbit Ga is closed we have

$$\mathcal{S}^*(N_{Ga,a}^X)^{G_a,\chi} = 0.$$

Then $\mathcal{S}^*(X)^{G,\chi} = 0$.



Conclusions

Conclusions

We reduce to the following

Task

Let a reductive group G act (linear) on an linear space V and let χ be a character of G.

Conclusions

We reduce to the following

Task

Let a reductive group G act (linear) on an linear space V and let χ be a character of G. We should prove that

$$\mathcal{S}^*(V)^{G,\chi} = 0$$

Conclusions

We reduce to the following

Task

Let a reductive group G act (linear) on an linear space V and let χ be a character of G. We should prove that

$$\mathcal{S}^*(V)^{G,\chi} = 0$$

- We may assume $V^G = 0$

Conclusions

We reduce to the following

Task

Let a reductive group G act (linear) on an linear space V and let χ be a character of G. We should prove that

$$\mathcal{S}^*(V)^{G,\chi} = 0$$

- We may assume $V^G = 0$
- Let $p : V \to V//G := spec(O(V)^G)$

Conclusions

We reduce to the following

Task

Let a reductive group G act (linear) on an linear space V and let χ be a character of G. We should prove that

$$\mathcal{S}^*(V)^{G,\chi} = 0$$

- We may assume $V^G = 0$
- Let $p : V \to V//G := spec(O(V)^G)$
- Let $\mathcal{N}(V) := p^{-1}(p(0))$

Conclusions

We reduce to the following

Task

Let a reductive group G act (linear) on an linear space V and let χ be a character of G. We should prove that

$$\mathcal{S}^*(V)^{G,\chi} = 0$$

- We may assume $V^G = 0$
- Let $p : V \to V//G := spec(O(V)^G)$
- Let $\mathcal{N}(V) := p^{-1}(p(0)) = \{x \in V | \overline{Gx} \ni 0\}$

Conclusions

We reduce to the following

Task

Let a reductive group G act (linear) on an linear space V and let χ be a character of G. We should prove that

$$\mathcal{S}^*(V)^{G,\chi} = 0$$

- We may assume $V^G = 0$
- Let $p : V \to V//G := spec(O(V)^G)$
- Let $\mathcal{N}(V) := p^{-1}(p(0)) = \{x \in V | \overline{Gx} \ni 0\}$
- Let $\mathcal{R}(V) := V - \mathcal{N}(V)$

Conclusions

We reduce to the following

Task

Let a reductive group G act (linear) on an linear space V and let χ be a character of G. We should prove that

$$\mathcal{S}^*(V)^{G,\chi} = 0$$

- We may assume $V^G = 0$
- Let $p : V \to V//G := spec(O(V)^G)$
- Let $\mathcal{N}(V) := p^{-1}(p(0)) = \{x \in V | \overline{Gx} \ni 0\}$
- Let $\mathcal{R}(V) := V - \mathcal{N}(V)$

by induction we may assume:

$$\mathcal{S}^*(\mathcal{R}(V))^{G,\chi} = 0.$$

Stratification

Proposition

Let $U \subset X$ be an open G-invariant subset and $Z := X - U$. Suppose that $\mathcal{S}^*(U)^{G,\chi} = 0$ and $\mathcal{S}^*_X(Z)^{G,\chi} = 0$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.

Proposition

Let $U \subset X$ be an open G-invariant subset and $Z := X - U$. Suppose that $\mathcal{S}^*(U)^{G,\chi} = 0$ and $\mathcal{S}^*_X(Z)^{G,\chi} = 0$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.

Proof.

$0 \to \mathcal{S}^*_X(Z)^{G,\chi} \to \mathcal{S}^*(X)^{G,\chi} \to \mathcal{S}^*(U)^{G,\chi}$. □

Stratification

Proposition

Let $U \subset X$ be an open G-invariant subset and $Z := X - U$. Suppose that $\mathcal{S}^*(U)^{G,\chi} = 0$ and $\mathcal{S}^*_X(Z)^{G,\chi} = 0$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.

Proof.

$0 \to \mathcal{S}^*_X(Z)^{G,\chi} \to \mathcal{S}^*(X)^{G,\chi} \to \mathcal{S}^*(U)^{G,\chi}$. □

For ℓ-spaces, $\mathcal{S}^*_X(Z)^{G,\chi} \cong \mathcal{S}^*(Z)^{G,\chi}$.

Stratification

Proposition

Let $U \subset X$ be an open G-invariant subset and $Z := X - U$. Suppose that $\mathcal{S}^*(U)^{G,\chi} = 0$ and $\mathcal{S}^*_X(Z)^{G,\chi} = 0$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.

Proof.

$0 \to \mathcal{S}^*_X(Z)^{G,\chi} \to \mathcal{S}^*(X)^{G,\chi} \to \mathcal{S}^*(U)^{G,\chi}$. □

For ℓ-spaces, $\mathcal{S}^*_X(Z)^{G,\chi} \cong \mathcal{S}^*(Z)^{G,\chi}$.
For smooth manifolds, there is a slightly more complicated statement which takes into account transversal derivatives:

Stratification

Proposition

Let $U \subset X$ be an open G-invariant subset and $Z := X - U$. Suppose that $\mathcal{S}^*(U)^{G,\chi} = 0$ and $\mathcal{S}^*_X(Z)^{G,\chi} = 0$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.

Proof.

$0 \to \mathcal{S}^*_X(Z)^{G,\chi} \to \mathcal{S}^*(X)^{G,\chi} \to \mathcal{S}^*(U)^{G,\chi}$. □

For ℓ-spaces, $\mathcal{S}^*_X(Z)^{G,\chi} \cong \mathcal{S}^*(Z)^{G,\chi}$.
For smooth manifolds, there is a slightly more complicated statement which takes into account transversal derivatives:

$$gr_k(\mathcal{S}^*_X(Z)) = \mathcal{S}^*(Z, Sym^k(CN_Z^X))$$

Fourier transform

Let V be a finite dimensional vector space over F and Q be a non-degenerate quadratic form on V. Let $\widehat{\xi}$ denote the Fourier transform of ξ defined using Q.

Fourier transform

Let V be a finite dimensional vector space over F and Q be a non-degenerate quadratic form on V. Let $\widehat{\xi}$ denote the Fourier transform of ξ defined using Q.

Proposition

Let G act on V linearly and preserving Q. Let $\xi \in \mathcal{S}^*(V)^{G,\chi}$. Then $\widehat{\xi} \in \mathcal{S}^*(V)^{G,\chi}$.

Fourier transform

Let V be a finite dimensional vector space over F and Q be a non-degenerate quadratic form on V. Let $\widehat{\xi}$ denote the Fourier transform of ξ defined using Q.

Proposition

Let G act on V linearly and preserving Q. Let $\xi \in \mathcal{S}^*(V)^{G,\chi}$. Then $\widehat{\xi} \in \mathcal{S}^*(V)^{G,\chi}$.

We reduce to the following

Task

$$(\mathcal{S}^*_V(\mathcal{N}(V)) \cap \mathcal{F}(\mathcal{S}^*_V(\mathcal{N}(V))))^{G,\chi} = 0$$

Fourier transform and homogeneity

Fourier transform and homogeneity

- We call a distribution $\xi \in \mathcal{S}^*(V)$ **abs-homogeneous of degree** d if for any $t \in F^\times$,

$$h_t(\xi) = u(t)|t|^d \xi,$$

where h_t denotes the homothety action on distributions and u is some unitary character of $F^\times$.

Fourier transform and homogeneity

- We call a distribution $\xi \in \mathcal{S}^*(V)$ **abs-homogeneous of degree** d if for any $t \in F^\times$,
$$h_t(\xi) = u(t)|t|^d \xi,$$
where h_t denotes the homothety action on distributions and u is some unitary character of $F^\times$.

Theorem (Jacquet, Rallis, Schiffmann,...)

Assume F is ***non-archimedean****. Let $\xi \in \mathcal{S}_V^*(Z(Q))$ be s.t. $\widehat{\xi} \in \mathcal{S}_V^*(Z(Q))$. Then ξ is abs-homogeneous of degree $\frac{1}{2} dim V$.*

Fourier transform and homogeneity

- We call a distribution $\xi \in \mathcal{S}^*(V)$ **abs-homogeneous of degree** d if for any $t \in F^\times$,

$$h_t(\xi) = u(t)|t|^d\xi,$$

where h_t denotes the homothety action on distributions and u is some unitary character of $F^\times$.

Theorem (Jacquet, Rallis, Schiffmann,...)

*Assume F is **non-archimedean**. Let $\xi \in \mathcal{S}^*_V(Z(Q))$ be s.t. $\widehat{\xi} \in \mathcal{S}^*_V(Z(Q))$. Then ξ is abs-homogeneous of degree $\frac{1}{2}dimV$.*

Theorem (Archimedean homogeneity – A., Gourevitch)

Let F be any local field. Let $L \subset \mathcal{S}^*_V(Z(Q))$ be a non-zero linear subspace s. t. $\forall \xi \in L$ we have $\widehat{\xi} \in L$ and $Q\xi \in L$.
Then there exists a non-zero distribution $\xi \in L$ which is abs-homogeneous of degree $\frac{1}{2}dimV$ or of degree $\frac{1}{2}dimV + 1$.

Singular Support and Wave Front Set

Singular Support and Wave Front Set

To a distribution ξ on X one assigns two subsets of T^*X.

Singular Support and Wave Front Set

To a distribution ξ on X one assigns two subsets of T^*X.

Singular Support (=Characteristic variety)	Wave front set
Defined using D-modules	Defined using Fourier transform
Available only in the Archimedean case	Available in both cases

Singular Support and Wave Front Set

To a distribution ξ on X one assigns two subsets of T^*X.

Singular Support (=Characteristic variety)	Wave front set
Defined using D-modules	Defined using Fourier transform
Available only in the Archimedean case	Available in both cases

In the non-Archimedean case we define the singular support to be the Zariski closure of the wave front set.

Properties and the Integrability Theorem

Let X be a smooth algebraic variety.

- Let $\xi \in \mathcal{S}^*(X)$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))$, where $p_X : T^*X \to X$ is the projection.

Properties and the Integrability Theorem

Let X be a smooth algebraic variety.

- Let $\xi \in \mathcal{S}^*(X)$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))$, where $p_X : T^*X \to X$ is the projection.
- Let an algebraic group G act on X. Let $\xi \in \mathcal{S}^*(X)^{G,\chi}$. Then

$$SS(\xi) \subset \{(x,\phi) \in T^*X \,|\, \forall \alpha \in \mathfrak{g} \quad \phi(\alpha(x)) = 0\}.$$

Properties and the Integrability Theorem

Let X be a smooth algebraic variety.

- Let $\xi \in \mathcal{S}^*(X)$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))$, where $p_X : T^*X \to X$ is the projection.
- Let an algebraic group G act on X. Let $\xi \in \mathcal{S}^*(X)^{G,\chi}$. Then

$$SS(\xi) \subset \{(x,\phi) \in T^*X \,|\, \forall \alpha \in \mathfrak{g} \quad \phi(\alpha(x)) = 0\}.$$

- Let V be a linear space. Let $Z \subset V^*$ be a closed subvariety, invariant with respect to homotheties. Let $\xi \in \mathcal{S}^*(V)$. Suppose that $\mathrm{Supp}(\widehat{\xi}) \subset Z$. Then $SS(\xi) \subset V \times Z$.

Properties and the Integrability Theorem

Let X be a smooth algebraic variety.

- Let $\xi \in \mathcal{S}^*(X)$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))$, where $p_X : T^*X \to X$ is the projection.
- Let an algebraic group G act on X. Let $\xi \in \mathcal{S}^*(X)^{G,\chi}$. Then

$$SS(\xi) \subset \{(x, \phi) \in T^*X \,|\, \forall \alpha \in \mathfrak{g} \quad \phi(\alpha(x)) = 0\}.$$

- Let V be a linear space. Let $Z \subset V^*$ be a closed subvariety, invariant with respect to homotheties. Let $\xi \in \mathcal{S}^*(V)$. Suppose that $\mathrm{Supp}(\widehat{\xi}) \subset Z$. Then $SS(\xi) \subset V \times Z$.
- Integrability theorem:
  Let $\xi \in \mathcal{S}^*(X)$. Then $SS(\xi)$ is (weakly) coisotropic.

Coisotropic varieties

Coisotropic varieties

Definition

Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it M**-coisotropic** if the following equivalent conditions hold.

Coisotropic varieties

Definition

Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it *M*-**coisotropic** if the following equivalent conditions hold.

- At every smooth point $z \in Z$ we have $T_z Z \supset (T_z Z)^{\perp}$. Here, $(T_z Z)^{\perp}$ denotes the orthogonal complement to $T_z Z$ in $T_z M$ with respect to ω.

Coisotropic varieties

Definition

Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it ***M*-coisotropic** if the following equivalent conditions hold.

- At every smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^{\perp}$. Here, $(T_zZ)^{\perp}$ denotes the orthogonal complement to T_zZ in T_zM with respect to ω.
- The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.

Coisotropic varieties

Definition

Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it M**-coisotropic** if the following equivalent conditions hold.

- At every smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^{\perp}$. Here, $(T_zZ)^{\perp}$ denotes the orthogonal complement to T_zZ in T_zM with respect to ω.
- The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.

- Every non-empty coisotropic subvariety of M has dimension at least $\frac{\dim M}{2}$.

Symmetric pairs

- A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^{\theta}$.

Symmetric pairs

- A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^{\theta}$.
- We call (G, H, θ) **connected** if G/H is Zariski connected.

Symmetric pairs

- A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^\theta$.
- We call (G, H, θ) **connected** if G/H is Zariski connected.
- Define an antiinvolution $\sigma : G \to G$ by $\sigma(g) := \theta(g^{-1})$.

Question

What symmetric pairs are Gelfand pairs?

Question

What symmetric pairs are Gelfand pairs?

For symmetric pairs of rank one this question was studied extensively by van-Dijk, Bosman, Rader and Rallis.

Question

What symmetric pairs are Gelfand pairs?

For symmetric pairs of rank one this question was studied extensively by van-Dijk, Bosman, Rader and Rallis.

Task

$$\mathcal{S}^*(G)^{H\times H} \subset \mathcal{S}^*(G)^{\sigma}$$

Question

What symmetric pairs are Gelfand pairs?

For symmetric pairs of rank one this question was studied extensively by van-Dijk, Bosman, Rader and Rallis.

Task

$$\mathcal{S}^*(G)^{H\times H} \subset \mathcal{S}^*(G)^{\sigma}$$

Necessary condition:

Definition

A symmetric pair (G, H, θ) is called **good** if σ preserves all closed $H \times H$ double closets.

Question

What symmetric pairs are Gelfand pairs?

For symmetric pairs of rank one this question was studied extensively by van-Dijk, Bosman, Rader and Rallis.

Task

$$\mathcal{S}^*(G)^{H\times H} \subset \mathcal{S}^*(G)^{\sigma}$$

Necessary condition:

Definition

A symmetric pair (G, H, θ) is called **good** if σ preserves all closed $H \times H$ double closets.

Proposition

Any connected symmetric pair over $\mathbb{C}$ is good.

Question

What symmetric pairs are Gelfand pairs?

For symmetric pairs of rank one this question was studied extensively by van-Dijk, Bosman, Rader and Rallis.

Task

$$\mathcal{S}^*(G)^{H\times H} \subset \mathcal{S}^*(G)^{\sigma}$$

Necessary condition:

Definition

A symmetric pair (G, H, θ) is called **good** if σ preserves all closed $H \times H$ double closets.

Proposition

Any connected symmetric pair over $\mathbb{C}$ is good.

Conjecture

Any good symmetric pair is a Gelfand pair.

How to complete the task?

How to complete the task?

Reformulate our task

How to complete the task?

Reformulate our task

Task

How to complete the task?

Reformulate our task

Task

Let $\widetilde{H \times H} = H \times H \rtimes \{1, \sigma\}$

How to complete the task?

Reformulate our task

Task

Let $\widetilde{H \times H} = H \times H \rtimes \{1, \sigma\}$ *and* $\chi : \widetilde{H \times H} \to \mathbb{C}$

How to complete the task?

Reformulate our task

Task

Let $\widetilde{H \times H} = H \times H \rtimes \{1, \sigma\}$ *and* $\chi : \widetilde{H \times H} \to \mathbb{C}$ *defaind by* $\chi(\widetilde{H \times H} - H \times H) = -1$

How to complete the task?

Reformulate our task

Task

Let $\widetilde{H \times H} = H \times H \rtimes \{1, \sigma\}$ *and* $\chi : \widetilde{H \times H} \to \mathbb{C}$ *defaind by* $\chi(\widetilde{H \times H} - H \times H) = -1$ *we have to show that*

How to complete the task?

Reformulate our task

Task

Let $\widetilde{H \times H} = H \times H \rtimes \{1, \sigma\}$ *and* $\chi : \widetilde{H \times H} \to \mathbb{C}$ *defaind by* $\chi(\widetilde{H \times H} - H \times H) = -1$ *we have to show that*

$$\mathcal{S}^*(G)^{\widetilde{H \times H}, \chi} = 0$$

How to complete the task?

Reformulate our task

Task

Let $\widetilde{H \times H} = H \times H \rtimes \{1, \sigma\}$ *and* $\chi : \widetilde{H \times H} \to \mathbb{C}$ *defaind by* $\chi(\widetilde{H \times H} - H \times H) = -1$ *we have to show that*

$$\mathcal{S}^*(G)^{\widetilde{H \times H}, \chi} = 0$$

Using Harsh-Chandra Descent it is enough to show that

How to complete the task?

Reformulate our task

Task

Let $\widetilde{H \times H} = H \times H \rtimes \{1, \sigma\}$ *and* $\chi : \widetilde{H \times H} \to \mathbb{C}$ *defaind by* $\chi(\widetilde{H \times H} - H \times H) = -1$ *we have to show that*

$$\mathcal{S}^*(G)^{\widetilde{H \times H}, \chi} = 0$$

Using Harsh-Chandra Descent it is enough to show that

1. The pair (G, H) is good

How to complete the task?

Reformulate our task

Task

Let $\widetilde{H \times H} = H \times H \rtimes \{1, \sigma\}$ *and* $\chi : \widetilde{H \times H} \to \mathbb{C}$ *defaind by* $\chi(\widetilde{H \times H} - H \times H) = -1$ *we have to show that*

$$\mathcal{S}^*(G)^{\widetilde{H \times H}, \chi} = 0$$

Using Harsh-Chandra Descent it is enough to show that

1. The pair (G, H) is good
2. $\mathcal{S}^*(\mathfrak{g}^\sigma)^{\widetilde{H}, \chi} = 0$ provided that $\mathcal{S}^*(\mathcal{R}(\mathfrak{g}^\sigma))^{\widetilde{H}, \chi} = 0$.

How to complete the task?

Reformulate our task

Task

Let $\widetilde{H \times H} = H \times H \rtimes \{1, \sigma\}$ *and* $\chi : \widetilde{H \times H} \to \mathbb{C}$ *defaind by* $\chi(\widetilde{H \times H} - H \times H) = -1$ *we have to show that*

$$\mathcal{S}^*(G)^{\widetilde{H \times H}, \chi} = 0$$

Using Harsh-Chandra Descent it is enough to show that

1. The pair (G, H) is good
2. $\mathcal{S}^*(\mathfrak{g}^\sigma)^{\widetilde{H}, \chi} = 0$ provided that $\mathcal{S}^*(\mathcal{R}(\mathfrak{g}^\sigma))^{\widetilde{H}, \chi} = 0$.
3. Compute all the "descendants" of the pair and prove (2) for them.

How to complete the task?

Reformulate our task

Task

Let $\widetilde{H \times H} = H \times H \rtimes \{1, \sigma\}$ *and* $\chi : \widetilde{H \times H} \to \mathbb{C}$ *defaind by* $\chi(\widetilde{H \times H} - H \times H) = -1$ *we have to show that*

$$\mathcal{S}^*(G)^{\widetilde{H \times H}, \chi} = 0$$

Using Harsh-Chandra Descent it is enough to show that

1. The pair (G, H) is good
2. $\mathcal{S}^*(\mathfrak{g}^\sigma)^{\widetilde{H}, \chi} = 0$ provided that $\mathcal{S}^*(\mathcal{R}(\mathfrak{g}^\sigma))^{\widetilde{H}, \chi} = 0$.
3. Compute all the "descendants" of the pair and prove (2) for them.

We call the property (2) regularity. We conjecture that all symmetric pairs are regular. This will imply that any good symmetric pair is a Gelfand pair.

How to prove regularity?

How to prove regularity?

- it is enough to prove that $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma)) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{\widetilde{H},\chi} = 0$

How to prove regularity?

- it is enough to prove that $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma)) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{\widetilde{H},\chi} = 0$
- Let $H' = \widetilde{H} \times F^\times$ and $\chi' = \chi \times |\cdot|^{\dim(\mathfrak{g}^\sigma)/2(+1)} u$

How to prove regularity?

- it is enough to prove that $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma)) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{\widetilde{H},\chi} = 0$
- Let $H' = \widetilde{H} \times F^\times$ and $\chi' = \chi \times |\cdot|^{\dim(\mathfrak{g}^\sigma)/2(+1)}u$
  Using Homoginity theorem it is enough to prove that: $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) = 0$

How to prove regularity?

- it is enough to prove that $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma)) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{\widetilde{H},\chi} = 0$
- Let $H' = \widetilde{H} \times F^\times$ and $\chi' = \chi \times |\cdot|^{\dim(\mathfrak{g}^\sigma)/2(+1)}u$
  Using Homoginity theorem it is enough to prove that: $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) = 0$
- We call an element $a \in \mathcal{N}(\mathfrak{g}^\sigma)$ distinguished if $\mathfrak{h}_a$ is nilpotent.

How to prove regularity?

- it is enough to prove that $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma)) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{\widetilde{H},\chi} = 0$
- Let $H' = \widetilde{H} \times F^\times$ and $\chi' = \chi \times |\cdot|^{\dim(\mathfrak{g}^\sigma)/2(+1)} u$
  Using Homoginity theorem it is enough to prove that: $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) = 0$
- We call an element $a \in \mathcal{N}(\mathfrak{g}^\sigma)$ distinguished if $\mathfrak{h}_a$ is nilpotent. Using Integrability theorem it is enough to prove that: $\mathcal{S}^*_{\mathfrak{g}^\sigma}(O)^{H',\chi'} = 0$ for any distinguished orbit O

How to prove regularity?

- it is enough to prove that
  $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma)) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{\widetilde{H},\chi} = 0$
- Let $H' = \widetilde{H} \times F^\times$ and $\chi' = \chi \times |\cdot|^{\dim(\mathfrak{g}^\sigma)/2(+1)} u$
  Using Homoginity theorem it is enough to prove that:
  $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) = 0$
- We call an element $a \in \mathcal{N}(\mathfrak{g}^\sigma)$ distinguished if $\mathfrak{h}_a$ is nilpotent. Using Integrability theorem it is enough to prove that: $\mathcal{S}^*_{\mathfrak{g}^\sigma}(O)^{H',\chi'} = 0$ for any distinguished orbit O
- Using Frobenius descent it is enough to prove that for any distinguished a:

How to prove regularity?

- it is enough to prove that $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma)) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{\widetilde{H},\chi} = 0$
- Let $H' = \widetilde{H} \times F^\times$ and $\chi' = \chi \times |\cdot|^{\dim(\mathfrak{g}^\sigma)/2(+1)}u$
  Using Homoginity theorem it is enough to prove that: $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) = 0$
- We call an element $a \in \mathcal{N}(\mathfrak{g}^\sigma)$ distinguished if $\mathfrak{h}_a$ is nilpotent. Using Integrability theorem it is enough to prove that: $\mathcal{S}^*_{\mathfrak{g}^\sigma}(O)^{H',\chi'} = 0$ for any distinguished orbit O
- Using Frobenius descent it is enough to prove that for any distinguished a:
  - $\chi'|_{H'_a}\Delta \neq 1$ – in the non-Archimidian case

How to prove regularity?

- it is enough to prove that $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma)) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{\widetilde{H},\chi} = 0$
- Let $H' = \widetilde{H} \times F^\times$ and $\chi' = \chi \times |\cdot|^{\dim(\mathfrak{g}^\sigma)/2(+1)} u$
  Using Homoginity theorem it is enough to prove that: $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) = 0$
- We call an element $a \in \mathcal{N}(\mathfrak{g}^\sigma)$ distinguished if $\mathfrak{h}_a$ is nilpotent. Using Integrability theorem it is enough to prove that: $\mathcal{S}^*_{\mathfrak{g}^\sigma}(O)^{H',\chi'} = 0$ for any distinguished orbit O
- Using Frobenius descent it is enough to prove that for any distinguished a:
  - $\chi'|_{H'_a\Delta} \neq 1$ – in the non-Archimidian case
  - $(N^{\mathfrak{g}^\sigma}_{Ha,a})^{H'_a,\chi'\Delta} = 0$ – in the Archimidian case

How to prove regularity?

- it is enough to prove that $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma)) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{\widetilde{H},\chi} = 0$
- Let $H' = \widetilde{H} \times F^\times$ and $\chi' = \chi \times |\cdot|^{\dim(\mathfrak{g}^\sigma)/2(+1)} u$
  Using Homoginity theorem it is enough to prove that: $(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) \cap \mathcal{F}(\mathcal{S}^*_{\mathfrak{g}^\sigma}(\mathcal{N}(\mathfrak{g}^\sigma))^{H',\chi'}) = 0$
- We call an element $a \in \mathcal{N}(\mathfrak{g}^\sigma)$ distinguished if $\mathfrak{h}_a$ is nilpotent. Using Integrability theorem it is enough to prove that: $\mathcal{S}^*_{\mathfrak{g}^\sigma}(O)^{H',\chi'} = 0$ for any distinguished orbit O
- Using Frobenius descent it is enough to prove that for any distinguished a:
  - $\chi'|_{H'_a}\Delta \neq 1$ – in the non-Archimidian case
  - $(N^{\mathfrak{g}^\sigma}_{Ha,a})^{H'_a,\chi'\Delta} = 0$ – in the Archimidian case

Regular symmetric pairs

Pair	p-adic case by	real case by
$(G \times G, \Delta G)$	A.-Gourevitch	A.-Gourevitch
$(GL_n(E), GL_n(F))$	Flicker	
$(GL_{n+k}, GL_n \times GL_k)$	Jacquet-Rallis	
$(O_{n+k}, O_n \times O_k)$	A.-Gourevitch	
(GL_n, O_n)		
(GL_{2n}, Sp_{2n})	Heumos - Rallis	A.-Sayag
$(sp_{2m}, sl_m \oplus \mathfrak{g}_a)$	A.	Sayag (based on work of Sekiguchi)
(e_6, sp_8)		
$(e_6, sl_6 \oplus sl_2)$		
(e_7, sl_8)		
(e_8, so_{16})		
$(f_4, sp_6 \oplus sl_2)$		
$(g_2, sl_2 \oplus sl_2)$		

Results on Gelfand pairs

<table>
<tr><th>Pair</th><th>p-adic case</th><th>real case</th></tr>
<tr><td>$(GL_n(E), GL_n(F))$</td><td>Flicker</td><td rowspan="4">A.-Gourevitch</td></tr>
<tr><td>$(GL_{n+k}, GL_n \times GL_k)$</td><td>Jacquet-Rallis</td></tr>
<tr><td>$(O_{n+k}, O_n \times O_k)$ over $\mathbb{C}$</td><td rowspan="2">——————</td></tr>
<tr><td>(GL_n, O_n) over $\mathbb{C}$</td></tr>
<tr><td>(GL_{2n}, Sp_{2n})</td><td>Heumos-Rallis</td><td>A.-Sayag</td></tr>
</table>

- real: $\mathbb{R}$ and $\mathbb{C}$
- p-adic: $\mathbb{Q}_p$ and its finite extensions.

Results on strong Gelfand pairs

Pair	p-adic	real
(GL_{n+1}, GL_n)	A.-Gourevitch-Rallis-Schiffmann	A.-Gourevitch, Sun-Zhu
$(O(V \oplus F), O(V))$		Sun-Zhu
$(U(V \oplus F), U(V))$		

- real: $\mathbb{R}$ and $\mathbb{C}$
- p-adic: $\mathbb{Q}_p$ and its finite extensions.

Results on strong Gelfand pairs

<table>
<tr><th>Pair</th><th>p-adic</th><th>real</th></tr>
<tr><td>(GL_{n+1}, GL_n)</td><td rowspan="3">A.-Gourevitch-Rallis-Schiffmann</td><td>A.-Gourevitch, Sun-Zhu</td></tr>
<tr><td>$(O(V \oplus F), O(V))$</td><td rowspan="2">Sun-Zhu</td></tr>
<tr><td>$(U(V \oplus F), U(V))$</td></tr>
</table>

- real: $\mathbb{R}$ and $\mathbb{C}$
- p-adic: $\mathbb{Q}_p$ and its finite extensions.

Remark

The results from the last two slides are used to prove splitting of periods of automorphic forms.