Holonomicity of spherical characters and applications to multiplicity bounds

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A distribution (or a D-module) ξ is called holonomic if

$$\dim(SS(\xi)) = \dim X$$

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Corollary

Let (π, V) be an admissible representation of $G(\mathbb{R})$ and $v_1 \in (V^*)^{H_1}$, $v_2 \in (\tilde{V}^*)^{H_2}$. Let ξ be the corresponding spherical character:

$$\langle \xi, f \rangle := \langle \pi^*(f) v_1, v_2 \rangle.$$

Then ξ is a holonomic distribution.

applications to the spherical character

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Let F be a local field. Then the wave front set of the spherical character of an admissible representation of $G(F)$ is included in a conic Lagrangian subvariety.

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Corollary

The spherical character of an admissible representation of $G(F)$ is smooth in a (Zariski) open dens set.

Bernstein-Kashiwara theorem

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Let X be a real algebraic manifold. Let M be a holonomic right D-module. Then $\dim Hom(M, \mathcal{S}^*(X)) < \infty$.

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Theorem (Bernstein, Kashiwara, A., Gourevitch, Minchenko)

Let X, Y be smooth algebraic varieties and $\mathcal{M}$ be a family of D_X-modules parameterized by Y. Suppose that $\mathcal{M}_y$ is holonomic. Then $\dim Hom(\mathcal{M}_y, \mathcal{S}^*(X))$ *is bounded when y ranges over Y.*

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Corollary (A., Gourevitch, Minchenko)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let $\mathcal{E}$ be an algebraic G-equivariant bundle on X. Then,

$$\dim \mathcal{S}^*(X, \mathcal{E})^{\mathfrak{g}, \chi} < \infty.$$

Moreover, it is bounded when we tensor $\mathcal{E}$ with a representation of $\mathfrak{g}$ of a fixed dimension.

Applications to multiplicities

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1. *If H is a spherical subgroup then there exists $C \in \mathbb{N}$ such that $\dim(\pi^*)^{\mathfrak{h},\chi} \leq C$ for any $\pi \in Irr(G)$ and any character χ of $\mathfrak{h}$.*
2. *If H is a real spherical subgroup then, for every irreducible admissible representation $\pi \in Irr(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n-dimensional representation τ of $\mathfrak{h}$ we have*

$$\dim Hom_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

Geometric formulation

Theorem (A., Gourevitch, Minchenko 2015)

Let

$$S = \{g \in G, x \in \mathfrak{g}^* | x \in \mathfrak{h}_1^\perp, ad(g)(x) \in \mathfrak{h}_2^\perp, x \text{ is nilpotent}\} =$$
$$= G \times \mathcal{N} \cap \bigcup_{g \in G} CN^G_{H_1 g H_2, g}$$

Then

$$\dim S = \dim G$$

The group case

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Translating the problem to $H = G/H$ we obtain:

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So

$$S' \subset \bigcup_{x \in \mathcal{N}_H} CN^{\mathfrak{h}}_{ad(G)x,x}$$

Springer resolution and Steinberg theorem

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Theorem (Steinberg 1976)

$$\dim G_\eta - 2\dim \mu^{-1}(\eta) = \mathrm{rk} G.$$

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T^*\mathcal{B} \times T^*\mathcal{B} & & G \times \mathcal{N} \\
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Where
$\mathcal{N}_{\mathfrak{h}_i} := \mathcal{N} \cap \mathfrak{h}_i^{\perp}$ and $L_i := \{(B, X) \in T^*\mathcal{B} \,|\, X \in \mathfrak{h}_i^{\perp}\}$

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$$T^*\mathcal{B} \times T^*\mathcal{B} \xrightarrow{\mu \times \mu} \mathcal{N} \times \mathcal{N} \xleftarrow{(g,x) \mapsto (x, \mathrm{Ad}^*(g^{-1})x)} G \times \mathcal{N}$$

$$\mathcal{N} \times \mathcal{N} \xrightarrow{res} \mathfrak{h}_1^* \times \mathfrak{h}_2^*$$

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$$L_1 \times L_2 \longrightarrow \mathcal{N}_{\mathfrak{h}_1} \times \mathcal{N}_{\mathfrak{h}_2} \longleftarrow S$$

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The estimate on dim S follows from the Stenberg theorem and:

$$\dim L_i = \dim \mathcal{B}$$