

Holonomicity of spherical characters and applications to multiplicity bounds

A. Aizenbud

Weizmann Institute of Science

Joint with: Dmitry Gourevitch and Andrey Minchenko

<http://www.wisdom.weizmann.ac.il/~aizenr/>

Holonomic D-modules and distributions

Holonomic D -modules and distributions

A D -module over a smooth affine algebraic variety X is a module over the ring of differential operators $D(X)$ on X .

Holonomic D -modules and distributions

A D -module over a smooth affine algebraic variety X is a module over the ring of differential operators $D(X)$ on X . A D -module M given by generators and relations can be thought of as a system of PDE.

Holonomic D -modules and distributions

A D -module over a smooth affine algebraic variety X is a module over the ring of differential operators $D(X)$ on X . A D -module M given by generators and relations can be thought of as a system of PDE. A solution of M is a D -modules homomorphism of M to an appropriate space of functions.

Holonomic D -modules and distributions

A D -module over a smooth affine algebraic variety X is a module over the ring of differential operators $D(X)$ on X . A D -module M given by generators and relations can be thought of as a system of PDE. A solution of M is a D -modules homomorphism of M to an appropriate space of functions.

Definition

Let M be a D -module over X with generators $m_1 \dots m_k$.

Holonomic D-modules and distributions

A D -module over a smooth affine algebraic variety X is a module over the ring of differential operators $D(X)$ on X . A D -module M given by generators and relations can be thought of as a system of PDE. A solution of M is a D -modules homomorphism of M to an appropriate space of functions.

Definition

Let M be a D -module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree i and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$.

Holonomic D-modules and distributions

A D -module over a smooth affine algebraic variety X is a module over the ring of differential operators $D(X)$ on X . A D -module M given by generators and relations can be thought of as a system of PDE. A solution of M is a D -modules homomorphism of M to an appropriate space of functions.

Definition

Let M be a D -module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree i and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$. Define

$$SS(M) = \text{supp}(gr_F(M)) \subset T^*(X)$$

Holonomic D-modules and distributions

A D -module over a smooth affine algebraic variety X is a module over the ring of differential operators $D(X)$ on X . A D -module M given by generators and relations can be thought of as a system of PDE. A solution of M is a D -modules homomorphism of M to an appropriate space of functions.

Definition

Let M be a D -module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree i and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$. Define

$$SS(M) = \text{supp}(gr_F(M)) \subset T^*(X)$$

For a distribution ξ on $X(\mathbb{R})$ define

$$SS(\xi) = SS(D(X)\xi)$$

Holonomic D-modules and distributions

A D -module over a smooth affine algebraic variety X is a module over the ring of differential operators $D(X)$ on X . A D -module M given by generators and relations can be thought of as a system of PDE. A solution of M is a D -modules homomorphism of M to an appropriate space of functions.

Definition

Let M be a D -module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree i and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$. Define

$$SS(M) = \text{supp}(gr_F(M)) \subset T^*(X)$$

For a distribution ξ on $X(\mathbb{R})$ define

$$SS(\xi) = SS(D(X)\xi) = \bigcap_{d\xi=0} \text{Zeros}(\text{symbol}(d))$$

Holonomic D-modules and distributions

A D -module over a smooth affine algebraic variety X is a module over the ring of differential operators $D(X)$ on X . A D -module M given by generators and relations can be thought of as a system of PDE. A solution of M is a D -modules homomorphism of M to an appropriate space of functions.

Definition

Let M be a D -module over X with generators $m_1 \dots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree i and $F_i(M) := F_i(D(X))(m_1 \dots m_k)$. Define

$$SS(M) = \text{supp}(gr_F(M)) \subset T^*(X)$$

For a distribution ξ on $X(\mathbb{R})$ define

$$SS(\xi) = SS(D(X)\xi) = \bigcap_{d\xi=0} \text{Zeros}(\text{symbol}(d))$$

A distribution (or a D -module) ξ is called holonomic if

$$\dim(SS(\xi)) = \dim X$$

Main results

Theorem (A., Gourevitch, Minchenko 2015)

Let G be an algebraic reductive group $H_i \subset G$ be spherical subgroups (i.e. $H_i B$ is open). The following system of equations on a distribution ξ on G is holonomic:

Theorem (A., Gourevitch, Minchenko 2015)

Let G be an algebraic reductive group $H_i \subset G$ be spherical subgroups (i.e. $H_i B$ is open). The following system of equations on a distribution ξ on G is holonomic:

- ξ is left H_1 invariant

Theorem (A., Gourevitch, Minchenko 2015)

Let G be an algebraic reductive group $H_i \subset G$ be spherical subgroups (i.e. $H_i B$ is open). The following system of equations on a distribution ξ on G is holonomic:

- ξ is left H_1 invariant
- ξ is right H_2 invariant

Theorem (A., Gourevitch, Minchenko 2015)

Let G be an algebraic reductive group $H_i \subset G$ be spherical subgroups (i.e. $H_i B$ is open). The following system of equations on a distribution ξ on G is holonomic:

- ξ is left H_1 invariant
- ξ is right H_2 invariant
- ξ is eigen w.r.t. the center $\mathfrak{z}(U(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of G .

Theorem (A., Gourevitch, Minchenko 2015)

Let G be an algebraic reductive group $H_i \subset G$ be spherical subgroups (i.e. $H_i B$ is open). The following system of equations on a distribution ξ on G is holonomic:

- ξ is left H_1 invariant
- ξ is right H_2 invariant
- ξ is eigen w.r.t. the center $\mathfrak{z}(u(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of G .

Corollary

Let (π, V) be an admissible representation of $G(\mathbb{R})$ and $v_1 \in (V^*)^{H_1}$, $v_2 \in (\tilde{V}^*)^{H_2}$. Let ξ be the corresponding spherical character:

$$\langle \xi, f \rangle := \langle \pi^*(f) v_1, v_2 \rangle.$$

Then ξ is a holonomic distribution.

applications to the spherical character

Corollary (A., Gourevitch, Minchenko, Sayag)

Let F be a local field. Then the wave front set of the spherical character of an admissible representation of $G(F)$ is included in a conic Lagrangian subvariety.

Corollary (A., Gourevitch, Minchenko, Sayag)

Let F be a local field. Then the wave front set of the spherical character of an admissible representation of $G(F)$ is included in a conic Lagrangian subvariety.

Corollary

The spherical character of an admissible representation of $G(F)$ is smooth in a (Zariski) open dens set.

Bernstein-Kashiwara theorem

Theorem (Bernstein, Kashiwara ~1974)

Let X be a real algebraic manifold. Let M be a holonomic right D -module. Then $\dim \text{Hom}(M, \mathcal{S}^(X)) < \infty$.*

Bernstein-Kashiwara theorem

Theorem (Bernstein, Kashiwara ~1974)

Let X be a real algebraic manifold. Let M be a holonomic right D -module. Then $\dim \operatorname{Hom}(M, S^(X)) < \infty$.*

Theorem (Bernstein, Kashiwara, A., Gourevitch, Minchenko)

Let X, Y be smooth algebraic varieties and \mathcal{M} be a family of D_X -modules parameterized by Y . Suppose that \mathcal{M}_y is holonomic. Then $\dim \operatorname{Hom}(\mathcal{M}_y, S^(X))$ is bounded when y ranges over Y .*

Bernstein-Kashiwara theorem

Theorem (Bernstein, Kashiwara ~1974)

Let X be a real algebraic manifold. Let M be a holonomic right D -module. Then $\dim \operatorname{Hom}(M, S^(X)) < \infty$.*

Theorem (Bernstein, Kashiwara, A., Gourevitch, Minchenko)

Let X, Y be smooth algebraic varieties and \mathcal{M} be a family of D_X -modules parameterized by Y . Suppose that \mathcal{M}_y is holonomic. Then $\dim \operatorname{Hom}(\mathcal{M}_y, S^(X))$ is bounded when y ranges over Y .*

Corollary (A., Gourevitch, Minchenko)

Let a real algebraic group G act on a real algebraic manifold X with finitely many orbits. Let \mathcal{E} be an algebraic G -equivariant bundle on X . Then,

$$\dim S^*(X, \mathcal{E})^{\mathfrak{g}, X} < \infty.$$

Moreover, it is bounded when we tensor \mathcal{E} with a representation of \mathfrak{g} of a fixed dimension.

We reprove the following theorem

We reprove the following theorem

Theorem (Kobayashi, Krötz, Oshima, Schlichtkrull 2013)

G be a real reductive group, H be a Zariski closed subgroup, and \mathfrak{h} be the Lie algebra of H .

We reprove the following theorem

Theorem (Kobayashi, Krötz, Oshima, Schlichtkrull 2013)

G be a real reductive group, H be a Zariski closed subgroup, and \mathfrak{h} be the Lie algebra of H .

- 1 *If H is a spherical subgroup then there exists $C \in \mathbb{N}$ such that $\dim(\pi^*)^{\mathfrak{h}, \chi} \leq C$ for any $\pi \in \text{Irr}(G)$ and any character χ of \mathfrak{h} .*

We reprove the following theorem

Theorem (Kobayashi, Krötz, Oshima, Schlichtkrull 2013)

G be a real reductive group, H be a Zariski closed subgroup, and \mathfrak{h} be the Lie algebra of H .

- 1 If H is a spherical subgroup then there exists $C \in \mathbb{N}$ such that $\dim(\pi^*)^{\mathfrak{h}, \chi} \leq C$ for any $\pi \in \text{Irr}(G)$ and any character χ of \mathfrak{h} .*
- 2 If H is a real spherical subgroup then, for every irreducible admissible representation $\pi \in \text{Irr}(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every n -dimensional representation τ of \mathfrak{h} we have*

$$\dim \text{Hom}_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$

Theorem (A., Gourevitch, Minchenko 2015)

Let

$$\begin{aligned} S &= \{g \in G, x \in \mathfrak{g}^* \mid x \in \mathfrak{h}_1^\perp, ad(g)(x) \in \mathfrak{h}_2^\perp, x \text{ is nilpotent}\} = \\ &= G \times \mathcal{N} \cap \bigcup_{g \in G} CN_{H_1 g H_2, g}^G \end{aligned}$$

Then

$$\dim S = \dim G$$

The group case

Assume $H_1 = H_2 = H$, diagonally embedded in $G = H \times H$.

The group case

Assume $H_1 = H_2 = H$, diagonally embedded in $G = H \times H$.
Translating the problem to $H = G/H$ we obtain:

$$S' = \{g \in H, x \in \mathfrak{h}^* | Ad(g)(x) = x, x \in \mathcal{N}_H\} = H \times \mathcal{N}_H \cap \bigcup_{g \in H} CN_{ad(G)g,g}^H$$

The group case

Assume $H_1 = H_2 = H$, diagonally embedded in $G = H \times H$.
Translating the problem to $H = G/H$ we obtain:

$$S' = \{g \in H, x \in \mathfrak{h}^* | Ad(g)(x) = x, x \in \mathcal{N}_H\} = H \times \mathcal{N}_H \cap \bigcup_{g \in H} CN_{ad(G)g,g}^H$$

passing to the Lie algebra

$$S' = \{g \in \mathfrak{h}, x \in \mathfrak{h} | [x, g] = 0, x \text{ is nilpotent}\}$$

The group case

Assume $H_1 = H_2 = H$, diagonally embedded in $G = H \times H$.
Translating the problem to $H = G/H$ we obtain:

$$S' = \{g \in H, x \in \mathfrak{h}^* | Ad(g)(x) = x, x \in \mathcal{N}_H\} = H \times \mathcal{N}_H \cap \bigcup_{g \in H} CN_{ad(G)g,g}^H$$

passing to the Lie algebra

$$S' = \{g \in \mathfrak{h}, x \in \mathfrak{h} | [x, g] = 0, x \text{ is nilpotent}\}$$

So

$$S' \subset \bigcup_{x \in \mathcal{N}_H} CN_{ad(G)x,x}^{\mathfrak{h}}$$

Let \mathcal{B} be the flag variety.

Springer resolution and Steinberg theorem

Let \mathcal{B} be the flag variety. $T^*\mathcal{B} \cong \{B \in \mathcal{B}, x \in \mathfrak{b}^\perp\}$.

Springer resolution and Steinberg theorem

Let \mathcal{B} be the flag variety. $T^*\mathcal{B} \cong \{B \in \mathcal{B}, x \in \mathfrak{b}^\perp\}$. We have a natural map $\mu : T^*\mathcal{B} \rightarrow \mathcal{N}$. It is called the Springer resolution.

Springer resolution and Steinberg theorem

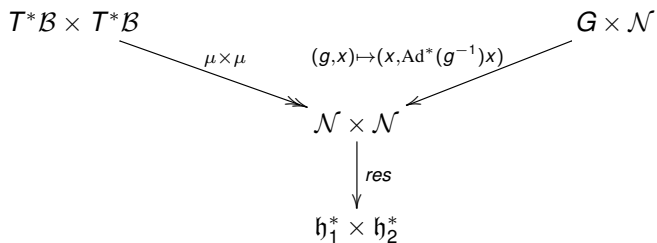
Let \mathcal{B} be the flag variety. $T^*\mathcal{B} \cong \{B \in \mathcal{B}, x \in \mathfrak{b}^\perp\}$. We have a natural map $\mu : T^*\mathcal{B} \rightarrow \mathcal{N}$. It is called the Springer resolution.

Theorem (Steinberg 1976)

$$\dim G_\eta - 2 \dim \mu^{-1}(\eta) = \text{rk} G.$$

Idea of the proof

Idea of the proof

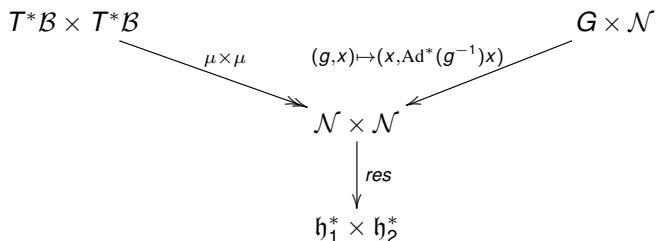


Idea of the proof

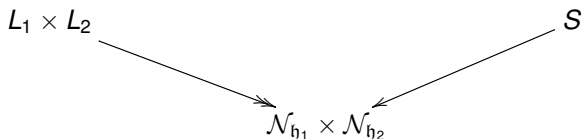
$$\begin{array}{ccc} T^*\mathcal{B} \times T^*\mathcal{B} & & G \times \mathcal{N} \\ & \begin{array}{c} \xrightarrow{\mu \times \mu} \\ \xrightarrow{(g,x) \mapsto (x, \text{Ad}^*(g^{-1})x)} \end{array} & \\ & \mathcal{N} \times \mathcal{N} & \\ & \downarrow \text{res} & \\ & \mathfrak{h}_1^* \times \mathfrak{h}_2^* & \end{array}$$

Passing to the fiber of $0 \in \mathfrak{h}_1^* \times \mathfrak{h}_2^*$ we get:

Idea of the proof



Passing to the fiber of $0 \in \mathfrak{h}_1^* \times \mathfrak{h}_2^*$ we get:



Where

$\mathcal{N}_{\mathfrak{h}_i} := \mathcal{N} \cap \mathfrak{h}_i^\perp$ and $L_i := \{(B, X) \in T^*\mathcal{B} \mid X \in \mathfrak{h}_i^\perp\}$

Idea of the proof

$$\begin{array}{ccc} T^*\mathcal{B} \times T^*\mathcal{B} & & G \times \mathcal{N} \\ & \searrow^{\mu \times \mu} \quad \swarrow^{(g,x) \mapsto (x, \text{Ad}^*(g^{-1})x)} & \\ & \mathcal{N} \times \mathcal{N} & \\ & \downarrow^{\text{res}} & \\ & \mathfrak{h}_1^* \times \mathfrak{h}_2^* & \end{array}$$

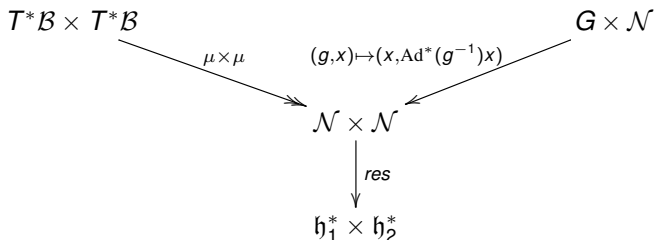
Passing to the fiber of $0 \in \mathfrak{h}_1^* \times \mathfrak{h}_2^*$ we get:

$$\begin{array}{ccc} L_1 \times L_2 & & S \\ & \searrow & \swarrow \\ & \mathcal{N}_{\mathfrak{h}_1} \times \mathcal{N}_{\mathfrak{h}_2} & \end{array}$$

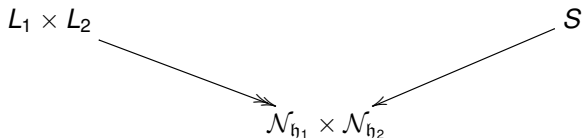
Where

$$\mathcal{N}_{\mathfrak{h}_i} := \mathcal{N} \cap \mathfrak{h}_i^\perp \text{ and } L_i := \{(B, X) \in T^*\mathcal{B} \mid X \in \mathfrak{h}_i^\perp\} = \bigcup_{x \in \mathcal{B}} \text{CN}_{H_i, x}^{\mathcal{B}}.$$

Idea of the proof



Passing to the fiber of $0 \in \mathfrak{h}_1^* \times \mathfrak{h}_2^*$ we get:



Where

$\mathcal{N}_{\mathfrak{h}_i} := \mathcal{N} \cap \mathfrak{h}_i^\perp$ and $L_i := \{(B, X) \in T^*\mathcal{B} \mid X \in \mathfrak{h}_i^\perp\} = \bigcup_{x \in \mathcal{B}} \text{CN}_{H_i x, x}^\mathcal{B}$.

The estimate on $\dim S$ follows from the Stenberg theorem and:

$$\dim L_i = \dim \mathcal{B}$$