Invariant Distributions and Gelfand Pairs

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Definition

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$$dimHom_H(\rho, \mathbb{C}) \leq 1.$$

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Theorem (Gelfand-Kazhdan,...)

Let σ be an involutive anti-automorphism of G (i.e. $\sigma(g_1 g_2) = \sigma(g_2)\sigma(g_1)$) and $\sigma^2 = Id$ and assume $\sigma(H) = H$. Suppose that $\sigma(\xi) = \xi$ for all bi H-invariant distributions ξ on G. Then (G, H) is a Gelfand pair.

Strong Gelfand Pairs

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A pair of groups (G, H) is called a **strong Gelfand pair** if for any irreducible "admissible" representations ρ of G and τ of H

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Proposition

The pair (G, H) is a strong Gelfand pair if and only if the pair $(G \times H, \Delta H)$ is a Gelfand pair.

Corollary

Let σ be an involutive anti-automorphism of G s.t. $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all distributions ξ on G invariant with respect to conjugation by H. Then (G, H) is a strong Gelfand pair.

Results

Local fields of characteristic zero:

- Archimedean: $\mathbb{R}$ and $\mathbb{C}$
- Non-archimedean(p-adic): $\mathbb{Q}_p$ and its finite extensions.

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<table>
<tr><th>Pair</th><th>Field</th><th>By</th></tr>
<tr><td>(GL_{n+1}, GL_n)</td><td rowspan="4">any</td><td>A.-G.-Sayag, van-Dijk</td></tr>
<tr><td>$(O(V \oplus F), O(V))$</td><td>van-Dijk-Bossmann-Aparicio, A.-G.-Sayag</td></tr>
<tr><td>$(GL_n(E), GL_n(F))$</td><td>Flicker, A.-G.</td></tr>
<tr><td>$(GL_{n+k}, GL_n \times GL_k)$</td><td>Jacquet-Rallis, A.-G.</td></tr>
<tr><td>$(O_{n+k}, O_n \times O_k)$</td><td rowspan="2">$\mathbb{C}$</td><td rowspan="2">A.-G.</td></tr>
<tr><td>(GL_n, O_n)</td></tr>
<tr><td>(GL_{2n}, Sp_{2n})</td><td>$F \neq \mathbb{R}$</td><td>Heumos - Rallis, Sayag</td></tr>
<tr><td rowspan="2">(GL_{n+1}, GL_n) strong</td><td>$\mathbb{R}$,$\mathbb{C}$</td><td>Aizenbud-Gourevitch</td></tr>
<tr><td rowspan="3">p-adic</td><td rowspan="3">Aizenbud-Gourevitch-
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Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M. We denote by $\mathcal{D}(M) := (C_c^\infty(M))^*$ the space of distributions on M. Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M.

Distributions on smooth manifolds and ℓ-spaces

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Definition

An ℓ-space is a Hausdorff locally compact totally disconnected topological space. For an ℓ-space X we denote by $\mathcal{S}(X)$ the space of compactly supported locally constant functions on X. We let $\mathcal{S}^*(X) := \mathcal{D}(X) := \mathcal{S}(X)^*$ be the space of distributions on X.

For a closed subset $Z \subset X$ we denote by $\mathcal{D}_X(Z)$ the space of distributions on X supported in Z.

Proposition

Let $Z \subset X$ be a closed subset and $U := X - Z$. Then we have the exact sequence

$$0 \to \mathcal{D}_X(Z) \to \mathcal{D}(X) \to \mathcal{D}(U).$$

Distributions supported in a closed subset

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For ℓ-spaces, $\mathcal{D}_X(Z) \cong \mathcal{D}(Z)$.
For smooth manifolds, $\mathcal{D}_X(Z)$ has an infinite filtration whose factors are $\mathcal{D}(Z, Sym^k(CN_Z^X))$, where $Sym^k(CN_Z^X)$ denote symmetric powers of the conormal bundle to Z.

Geometric conditions

Setting

Let G be an algebraic group over a local field F. Let H be a closed algebraic subgroup. Let $\sigma : G \to G$ be an antiinvolution. We want to show that every $H \times H$ invariant distribution on G is σ-invariant.

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A necessary condition for that is :
"σ preserves every closed double coset (which carries $H \times H$ invariant distribution)".

Over p-adic fields, it is sufficient (but not necessary) to prove that σ preserves every double coset.

Reformulation of the problem

Notation

Let σ *act on* $H \times H$ *by* $\sigma(h_1, h_2) := (\sigma(h_2^{-1}), \sigma(h_1^{-1}))$*. Denote*

$$\widetilde{H \times H} := (H \times H) \rtimes \{1, \sigma\}.$$

It has a natural action on G. Define a character χ *of* $\widetilde{H \times H}$ *by*

$$\chi(H \times H) = \{1\},\ \chi(\widetilde{H \times H} - (H \times H)) = \{-1\}.$$

Now our problem becomes equivalent to $\mathcal{D}(G)^{\widetilde{H \times H}, \chi} = 0$.

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Let $U \subset X$ be an open G-invariant subset and $Z := X - U$. Suppose that $\mathcal{D}(U)^{G,\chi} = 0$ and $\mathcal{D}_X(Z)^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$.

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Proof.

$0 \to \mathcal{D}_X(Z)^{G,\chi} \to \mathcal{D}(X)^{G,\chi} \to \mathcal{D}(U)^{G,\chi}$. □

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Proof.

$0 \to \mathcal{D}_X(Z)^{G,\chi} \to \mathcal{D}(X)^{G,\chi} \to \mathcal{D}(U)^{G,\chi}$. □

For ℓ-spaces, $\mathcal{D}_X(Z)^{G,\chi} \cong \mathcal{D}(Z)^{G,\chi}$.
For smooth manifolds, to show $\mathcal{D}_X(Z)^{G,\chi}$ it is enough to show that $\mathcal{D}(Z, Sym^k(CN_Z^X))^{G,\chi} = 0$ for any k.

Frobenius reciprocity

$$\begin{array}{ccc} X_z & \longrightarrow & X \\ \downarrow & & \downarrow \\ z & \longrightarrow & Z \end{array}$$

Theorem (Bernstein, Baruch, ...)

Let $\psi : X \to Z$ be a map.
Let a G act on X and Z such that $\psi(gx) = g\psi(x)$.
Suppose that the action of G on Z is transitive.
Suppose that both G and $Stab_G(z)$ are unimodular. Then

$$\mathcal{D}(X)^{G,\chi} \cong \mathcal{D}(X_z)^{Stab_G(z),\chi}.$$

Reductive groups

Example

GL_n, semisimple groups, O_n, U_n, Sp_{2n},...

Fact

Any algebraic representation of a reductive group decomposes to a direct sum of irreducible representations.

Fact

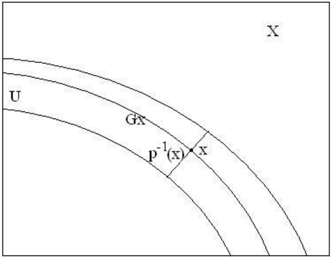
Reductive groups are unimodular.

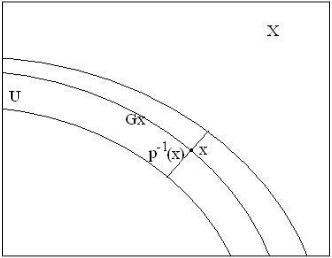
Luna's slice theorem

We say that $x \in X$ is G-semisimple if its orbit is closed.

Theorem (Luna's slice theorem)

Let a reductive group G act on a smooth affine algebraic variety X. Let $x \in X$ be G-semisimple. Then there exist
(i) an open G-invariant neighborhood U of Gx in X with a G-equivariant retract $p : U \to Gx$ and
(ii) a G_x-equivariant embedding $\psi : p^{-1}(x) \hookrightarrow N^X_{Gx,x}$ with open image such that $\psi(x) = 0$.





Theorem

Let a reductive group G act on a smooth affine algebraic variety X. Let χ be a character of G. Suppose that for any G-semisimple $x \in X$ we have

$$\mathcal{D}(N_{Gx,x}^{X})^{G_x,\chi} = 0.$$

Then $\mathcal{D}(X)^{G,\chi} = 0$.

A stronger version

Let V be an algebraic finite dimensional representation over F of a reductive group G.

- $Q(V) := (V/V^G)$. Since G is reductive, there is a canonical splitting $V = Q(V) \oplus V^G$.
- $\Gamma(V) := \{v \in Q(V) | \overline{Gv} \ni 0\}$.
- $R(V) := Q(V) - \Gamma(V)$.

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Theorem

Let a reductive group G act on a smooth affine variety X. Let χ be a character of G. Suppose that for any G-semisimple $x \in X$ such that

$$\mathcal{D}(R(N^X_{Gx,x}))^{G_x,\chi} = 0$$

we have

$$\mathcal{D}(Q(N^X_{Gx,x}))^{G_x,\chi} = 0.$$

Then $\mathcal{D}(X)^{G,\chi} = 0$.

Let V be a finite dimensional vector space over F and B be a non-degenerate quadratic form on V. Let $\widehat{\xi}$ denote the Fourier transform of ξ defined using B.

Proposition

Let G act on V linearly and preserving B. Let $\xi \in \mathcal{S}^*(V)^{G,\chi}$. Then $\widehat{\xi} \in \mathcal{S}^*(V)^{G,\chi}$.

Fourier transform and homogeneity

- We call a distribution $\xi \in \mathcal{S}^*(V)$ **abs-homogeneous of degree** d if for any $t \in F^\times$,
$$h_t(\xi) = u(t)|t|^d \xi,$$
where h_t denotes the homothety action on distributions and u is some unitary character of $F^\times$.

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Theorem (Jacquet, Rallis, Schiffmann,...)

Assume F is ***non-archimedean****. Let $\xi \in \mathcal{S}_V^*(Z(B))$ be s. t. $\widehat{\xi} \in \mathcal{S}_V^*(Z(B))$. Then ξ is abs-homogeneous of degree $\frac{1}{2}dimV$.*

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Theorem (archimedean homogeneity)

Let F be any local field. Let $L \subset \mathcal{S}^*_V(Z(B))$ be a non-zero linear subspace s. t. $\forall \xi \in L$ we have $\widehat{\xi} \in L$ and $B\xi \in L$.
Then there exists a non-zero distribution $\xi \in L$ which is abs-homogeneous of degree $\frac{1}{2}dim V$ or of degree $\frac{1}{2}dim V + 1$.

Localization principle

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \downarrow & & \downarrow \\ y & \longrightarrow & Y \end{array}$$

Theorem (Aizenbud-Gourevitch-Sayag)

Let a reductive group G act on a smooth affine variety X. Let Y be an algebraic variety and $\phi : X \to Y$ be an algebraic G-invariant map. Let χ be a character of G. Suppose that for any $y \in Y$ we have $\mathcal{D}_X(X_y)^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$.

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For ℓ-spaces, a stronger version of this principle was proven by J. Bernstein 30 years ago.

Symmetric pairs

- A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^{\theta}$.
- We call (G, H, θ) **connected** if G/H is Zariski connected.
- Define an antiinvolution $\sigma : G \to G$ by $\sigma(g) := \theta(g^{-1})$.

Symmetric Gelfand pairs

- A symmetric pair (G, H, θ) is called **good** if σ preserves all closed $H \times H$ double cosets.

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Any good symmetric pair is a Gelfand pair.

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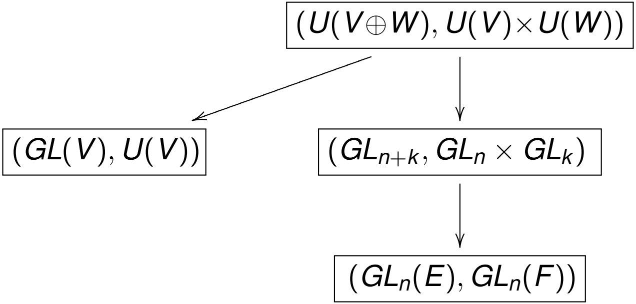
To check that a symmetric pair is Gelfand

1. Prove that it is good
2. Prove that there are no equivariant distributions supported on the singular set in the Lie algebra $\mathfrak{g}$.
3. Compute all the "descendants" of the pair and prove (2) for them.

Results for GL and U



Corollary

The pairs $(GL_n(E), GL_n(F))$ and $(GL_{n+k}, GL_n \times GL_k)$ are Gelfand pairs.

Results for O

$$(O(V \oplus W), O(V) \times O(W))$$

$$\downarrow$$

$$(U(V_E), O(V))$$

$$\downarrow$$

$$(GL(V), O(V))$$

Corollary

For $F = \mathbb{C}$, the pairs $(O(V \oplus W), O(V) \times O(W))$ and $(GL(V), O(V))$ are Gelfand pairs.

Results for non-symmetric pairs

Let F be a **p-adic** field. Then the following pairs are **strong** Gelfand pairs

$$(O(V \oplus F), O(V))$$

$$\downarrow$$

$$(U(V \oplus F), U(V))$$

$$\downarrow$$

$$(GL_{n+1}, GL_n)$$

Formulation

Let F be a p-adic field of characteristic zero.

Theorem (Aizenbud-Gourevitch-Rallis-Schiffmann)

Every $GL_n(F)$-invariant distribution on $GL_{n+1}(F)$ is transposition invariant.

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Reason:

$$g \begin{pmatrix} A_{n\times n} & v_{n\times 1} \\ \phi_{1\times n} & \lambda \end{pmatrix} g^{-1} = \begin{pmatrix} gAg^{-1} & gv \\ (g^*)^{-1}\phi & \lambda \end{pmatrix} \text{ and } \begin{pmatrix} A & v \\ \phi & \lambda \end{pmatrix}^t = \begin{pmatrix} A^t & \phi^t \\ v^t & \lambda \end{pmatrix}$$

Harish-Chandra descent

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- $\mathcal{N}_i := \{a \in \mathcal{N} | \dim Ga \leq i\} \subset \mathcal{N}$

We prove by descending induction on i that $\mathcal{S}^*(X)^{\widetilde{G},\chi} = \mathcal{S}^*(\mathcal{N}_i \times \Gamma)^{\widetilde{G},\chi}$.

Reduction

We assume $\mathcal{S}^*(X)^{\widetilde{G},\chi} = \mathcal{S}^*(\mathcal{N}_i \times \Gamma)^{\widetilde{G},\chi}$.
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Let $\xi \in \mathcal{S}^*(X)^{\widetilde{G},\chi}$. We know that for any λ,
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We have to show $\eta = 0$.

Key Lemma

It is enough to prove

Lemma (Key)

Any $\eta \in \mathcal{S}^*(O \times V \times V^*)^{\widetilde{G},\chi}$ *such that both* η *and* $\widehat{\eta}$ *are supported in* $\widetilde{O}$ *is zero.*

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Apply Frobenius reciprocity:

$$\begin{array}{ccc} \widetilde{O}_A & \longrightarrow & \widetilde{O} \\ \downarrow & & \downarrow \\ A & \longrightarrow & O \end{array}$$

- $A \in O$
- $\widetilde{O}_A := \{(v, \phi) \in V \times V^* | (A, v, \phi) \in \widetilde{O}\}$
- Let $G_A := Stab_G(A)$ denote the centralizer of A.
- $\widetilde{G}_A := Stab_{\widetilde{G}}(A)$

Reformulation

Equivalent formulation:

Lemma (Key')

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Reduction to Jordan block

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Proof.

$$\begin{pmatrix} v \\ w \end{pmatrix} \otimes \begin{pmatrix} \phi & \psi \end{pmatrix} = \begin{pmatrix} v \otimes \phi & * \\ * & w \otimes \psi \end{pmatrix}$$

$$[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}] = \begin{pmatrix} [A, X] & * \\ * & [B, W] \end{pmatrix}$$

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Hence we can assume that $A = J_n$ is one Jordan block.

Proof for Jordan block

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where $B(v,\phi) := \phi(v)$.
$\mathrm{Supp}(\zeta), \mathrm{Supp}(\widehat{\zeta}) \subset Z(B) \Rightarrow \zeta$ is abs-homogeneous of degree n.

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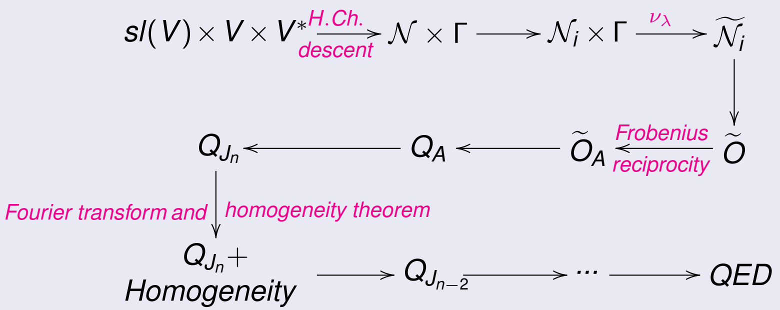
By induction $\zeta = 0$. $\square$

Flowchart


$sl(V) \times V \times V^*$
H.Ch.
descent
$\mathcal{N} \times \Gamma$
$\mathcal{N}_i \times \Gamma$
ν_λ
$\widetilde{\mathcal{N}_i}$
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...
QED

Let D be either F or a quadratic extension of F. Let V be a vector space over D. Let $< , >$ be a non-degenerate hermitian form on V. Let $W := V \oplus D$. Extend $< , >$ to W in the obvious way. Consider the embedding of $U(V)$ into $U(W)$.

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Theorem (Aizenbud-Gourevitch-Rallis-Schiffmann)

Every $U(V)$- invariant distribution on $U(W)$ is invariant with respect to transposition.

Orthogonal and unitary groups

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- $G := U(V)$
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- $X := su(V) \times V$
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Equivalent formulation:

Theorem

$\mathcal{S}^*(X)^{\widetilde{G},\chi} = 0.$

Sketch of the proof

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