Invariant Distributions and Gelfand Pairs

A. Aizenbud and D. Gourevitch

http://www.wisdom.weizmann.ac.il/~aizenr/
A pair of groups \((G \supset H)\) is called a **Gelfand pair** if for any irreducible "admissible" representation \(\rho\) of \(G\)

\[
\dim \text{Hom}_H(\rho, \mathbb{C}) \leq 1.
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Definition

A pair of groups $(G \supset H)$ is called a **Gelfand pair** if for any irreducible "admissible" representation $\rho$ of $G$

$$\dim \text{Hom}_H(\rho, \mathbb{C}) \leq 1.$$ 

Theorem (Gelfand-Kazhdan,...)

Let $\sigma$ be an involutive anti-automorphism of $G$ (i.e. $\sigma(g_1g_2) = \sigma(g_2)\sigma(g_1)$) and $\sigma^2 = \text{Id}$ and assume $\sigma(H) = H$. Suppose that $\sigma(\xi) = \xi$ for all bi $H$-invariant distributions $\xi$ on $G$. Then $(G, H)$ is a Gelfand pair.
A pair of groups \((G, H)\) is called a **strong Gelfand pair** if for any irreducible "admissible" representations \(\rho\) of \(G\) and \(\tau\) of \(H\)

\[
\dim \text{Hom}_H(\rho|_H, \tau) \leq 1.
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### Definition

A pair of groups $(G, H)$ is called a **strong Gelfand pair** if for any irreducible "admissible" representations $\rho$ of $G$ and $\tau$ of $H$

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### Proposition

*The pair $(G, H)$ is a strong Gelfand pair if and only if the pair $(G \times H, \Delta H)$ is a Gelfand pair.*
Strong Gelfand Pairs

Definition
A pair of groups \((G, H)\) is called a strong Gelfand pair if for any irreducible "admissible" representations \(\rho\) of \(G\) and \(\tau\) of \(H\)

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Proposition
The pair \((G, H)\) is a strong Gelfand pair if and only if the pair \((G \times H, \Delta H)\) is a Gelfand pair.

Corollary
Let \(\sigma\) be an involutive anti-automorphism of \(G\) s.t. \(\sigma(H) = H\). Suppose \(\sigma(\xi) = \xi\) for all distributions \(\xi\) on \(G\) invariant with respect to conjugation by \(H\). Then \((G, H)\) is a strong Gelfand pair.
Local fields of characteristic zero:
- Archimedean: \( \mathbb{R} \) and \( \mathbb{C} \)
- Non-archimedean(p-adic): \( \mathbb{Q}_p \) and its finite extensions.
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- **Archimedean**: $\mathbb{R}$ and $\mathbb{C}$
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<table>
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| $(GL_{2n}, Sp_{2n})$ | p-adic | Aizenbud-Gourevitch-
| $(GL_{n+1}, GL_n)$ strong | | -Rallis-Schiffmann |
| $(O(V \oplus F), O(V))$ strong | | |
| $(U(V \oplus F), U(V))$ strong | | |
Notation

Let $M$ be a smooth manifold. We denote by $C^\infty_c(M)$ the space of smooth compactly supported functions on $M$. We denote by $\mathcal{D}(M) := (C^\infty_c(M))^*$ the space of distributions on $M$. Sometimes we will also consider the space $S^*(M)$ of Schwartz distributions on $M$. 
Distributions on smooth manifolds and $\ell$-spaces

Notation

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Definition

An $\ell$-space is a Hausdorff locally compact totally disconnected topological space. For an $\ell$-space $X$ we denote by $S(X)$ the space of compactly supported locally constant functions on $X$. We let $S^*(X) := \mathcal{D}(X) := S(X)^*$ be the space of distributions on $X$. 
For a closed subset $Z \subset X$ we denote by $\mathcal{D}_X(Z)$ the space of distributions on $X$ supported in $Z$.

**Proposition**

Let $Z \subset X$ be a closed subset and $U := X - Z$. Then we have the exact sequence

$$0 \to \mathcal{D}_X(Z) \to \mathcal{D}(X) \to \mathcal{D}(U).$$
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For smooth manifolds, $\mathcal{D}_X(Z)$ has an infinite filtration whose factors are $\mathcal{D}(Z, \text{Sym}^k(\mathcal{CN}_Z^X))$, where $\text{Sym}^k(\mathcal{CN}_Z^X)$ denote symmetric powers of the conormal bundle to $Z$. 
Setting

Let $G$ be an algebraic group over a local field $F$. Let $H$ be a closed algebraic subgroup. Let $\sigma : G \to G$ be an antiinvolution. We want to show that every $H \times H$ invariant distribution on $G$ is $\sigma$-invariant.

A necessary condition for that is:

"$\sigma$ preserves every closed double coset (which carries $H \times H$ invariant distribution)."

Over p-adic fields, it is sufficient (but not necessary) to prove that $\sigma$ preserves every double coset.
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"$\sigma$ preserves every closed double coset (which carries $H \times H$ invariant distribution)".

Over p-adic fields, it is sufficient (but not necessary) to prove that $\sigma$ preserves every double coset.
Let $\sigma$ act on $H \times H$ by $\sigma(h_1, h_2) := (\sigma(h_2^{-1}), \sigma(h_1^{-1}))$. Denote $\widetilde{H \times H} := (H \times H) \rtimes \{1, \sigma\}$.

It has a natural action on $G$. Define a character $\chi$ of $\widetilde{H \times H}$ by

$$
\chi(H \times H) = \{1\}, \quad \chi(\widetilde{H \times H} - (H \times H)) = \{-1\}.
$$

Now our problem becomes equivalent to $\mathcal{D}(G)^{\widetilde{H \times H}, \chi} = 0$. 
First tool: Stratification

**Setting**

A group $G$ acts on a space $X$, and $\chi$ is a character of $G$. We want to show $D(X)^G,\chi = 0$. 

**Proposition**

Let $U \subset X$ be an open $G$-invariant subset and $Z := X \setminus U$. Suppose that $D(U)^G,\chi = 0$ and $D(X(Z))^{G,\chi} = 0$. Then $D(X)^G,\chi = 0$.

**Proof.**

$0 \rightarrow D(X(Z))^{G,\chi} \rightarrow D(X)^{G,\chi} \rightarrow D(U)^{G,\chi}$.

For $\ell$-spaces, $D(X(Z))^{G,\chi} \sim D(Z)^{G,\chi}$.

For smooth manifolds, to show $D(X(Z))^{G,\chi}$ it is enough to show that $D(Z,\text{Sym}^k(CN_X Z))^{G,\chi} = 0$ for any $k$. 

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Let $U \subset X$ be an open $G$-invariant subset and $Z := X - U$. Suppose that $\mathcal{D}(U)^{G,\chi} = 0$ and $\mathcal{D}_X(Z)^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$. 
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Proof.

$0 \to \mathcal{D}_X(Z)^{G,\chi} \to \mathcal{D}(X)^{G,\chi} \to \mathcal{D}(U)^{G,\chi}$. □
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Proof.

$0 \rightarrow \mathcal{D}_X(Z)^{G,\chi} \rightarrow \mathcal{D}(X)^{G,\chi} \rightarrow \mathcal{D}(U)^{G,\chi} \rightarrow 0$.

For $\ell$-spaces, $\mathcal{D}_X(Z)^{G,\chi} \cong \mathcal{D}(Z)^{G,\chi}$.

For smooth manifolds, to show $\mathcal{D}_X(Z)^{G,\chi}$ it is enough to show that $\mathcal{D}(Z, \text{Sym}^k(CN_X^Z))^G,\chi = 0$ for any $k$. 
Theorem (Bernstein, Baruch, ...)

Let $\psi : X \to Z$ be a map.
Let a $G$ act on $X$ and $Z$ such that $\psi(gx) = g\psi(x)$.
Suppose that the action of $G$ on $Z$ is transitive.
Suppose that both $G$ and $\text{Stab}_G(z)$ are unimodular. Then

$$D(X)^{G,\chi} \cong D(X_z)^{\text{Stab}_G(z),\chi}.$$
Example

$GL_n$, semisimple groups, $O_n$, $U_n$, $Sp_{2n}$, ...

Fact

Any algebraic representation of a reductive group decomposes to a direct sum of irreducible representations.

Fact

Reductive groups are unimodular.
We say that $x \in X$ is $G$-semisimple if its orbit is closed.

**Theorem (Luna’s slice theorem)**

Let a reductive group $G$ act on a smooth affine algebraic variety $X$. Let $x \in X$ be $G$-semisimple. Then there exist

(i) an open $G$-invariant neighborhood $U$ of $Gx$ in $X$ with a $G$-equivariant retract $p : U \to Gx$ and

(ii) a $G_x$-equivariant embedding $\psi : p^{-1}(x) \hookrightarrow N_{Gx,x}^X$ with open image such that $\psi(x) = 0$. 

[Diagram of the theorem]
Theorem

Let a reductive group $G$ act on a smooth affine algebraic variety $X$. Let $\chi$ be a character of $G$. Suppose that for any $G$-semisimple $x \in X$ we have

$$\mathcal{D}(N_{Gx,x}^X)^{G_{x},\chi} = 0.$$  

Then $\mathcal{D}(X)^{G,\chi} = 0$. 

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Generalized Harish-Chandra descent
Let $V$ be an algebraic finite dimensional representation over $F$ of a reductive group $G$.

- $Q(V) := (V/V^G)$. Since $G$ is reductive, there is a canonical splitting $V = Q(V) \oplus V^G$.
- $\Gamma(V) := \{v \in Q(V) | Gv \ni 0\}$.
- $R(V) := Q(V) - \Gamma(V)$. 

Theorem

Let a reductive group $G$ act on a smooth affine variety $X$. Let $\chi$ be a character of $G$. Suppose that for any $G$-semisimple $x \in X$ such that $D(Q(N_{X,G}x), x, \chi) = 0$ we have $D(Q(N_{X,G}x), x, \chi) = 0$. Then $D(X, G, \chi) = 0$.
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- $Q(V) := (V/V^G)$. Since $G$ is reductive, there is a canonical splitting $V = Q(V) \oplus V^G$.
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**Theorem**

Let a reductive group $G$ act on a smooth affine variety $X$. Let $\chi$ be a character of $G$. Suppose that for any $G$-semisimple $x \in X$ such that

$$\mathcal{D}(R(N_{Gx,x}^X))^{G_x,\chi} = 0$$

we have

$$\mathcal{D}(Q(N_{Gx,x}^X))^{G_x,\chi} = 0.$$

Then $\mathcal{D}(X)^{G,\chi} = 0$. 

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Invariant Distributions and Gelfand Pairs
Let $V$ be a finite dimensional vector space over $F$ and $B$ be a non-degenerate quadratic form on $V$. Let $\hat{\xi}$ denote the Fourier transform of $\xi$ defined using $B$.

**Proposition**

Let $G$ act on $V$ linearly and preserving $B$. Let $\xi \in S^*(V)^{G,\chi}$. Then $\hat{\xi} \in S^*(V)^{G,\chi}$.
We call a distribution $\xi \in S^*(V)$ abs-homogeneous of degree $d$ if for any $t \in F^\times$,

$$h_t(\xi) = u(t)|t|^d\xi,$$

where $h_t$ denotes the homothety action on distributions and $u$ is some unitary character of $F^\times$. 

---

Theorem (Jacquet, Rallis, Schiffmann,...)

Assume $F$ is non-archimedean. Let $\xi \in S^*(V)(\mathbb{Z}(B))$ be s. t. $\hat{\xi} \in S^*(V)(\mathbb{Z}(B))$. Then $\xi$ is abs-homogeneous of degree $\frac{1}{2}\dim V$.

Theorem (archimedean homogeneity)

Let $F$ be any local field. Let $L \subset S^*(V)(\mathbb{Z}(B))$ be a non-zero linear subspace s. t. $\forall \xi \in L$ we have $\hat{\xi} \in L$ and $B\xi \in L$. Then there exists a non-zero distribution $\xi \in L$ which is abs-homogeneous of degree $\frac{1}{2}\dim V$ or of degree $\frac{1}{2}\dim V + 1$. 
Fourier transform and homogeneity

- We call a distribution \( \xi \in S^*(V) \) abs-homogeneous of degree \( d \) if for any \( t \in F^\times \),

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Assume \( F \) is non-archimedean. Let \( \xi \in S^*_V(Z(B)) \) be s. t. \( \hat{\xi} \in S^*_V(Z(B)) \). Then \( \xi \) is abs-homogeneous of degree \( \frac{1}{2} \dim V \).
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Theorem (Aizenbud-Gourevitch-Sayag)

Let a reductive group $G$ act on a smooth affine variety $X$. Let $Y$ be an algebraic variety and $\phi : X \rightarrow Y$ be an algebraic $G$-invariant map. Let $\chi$ be a character of $G$. Suppose that for any $y \in Y$ we have $D^X(X_y)^{G,\chi} = 0$. Then $D^X(X)^{G,\chi} = 0$. 
Localization principle

Theorem (Aizenbud-Gourevitch-Sayag)

Let a reductive group $G$ act on a smooth affine variety $X$. Let $Y$ be an algebraic variety and $\phi : X \rightarrow Y$ be an algebraic $G$-invariant map. Let $\chi$ be a character of $G$. Suppose that for any $y \in Y$ we have $D_X(X_y)^G,\chi = 0$. Then $D(X)^G,\chi = 0$.

For $\ell$-spaces, a stronger version of this principle was proven by J. Bernstein 30 years ago.
Symmetric pairs

A **symmetric pair** is a triple \((G, H, \theta)\) where \(H \subset G\) are reductive groups, and \(\theta\) is an involution of \(G\) such that \(H = G^\theta\).

We call \((G, H, \theta)\) **connected** if \(G/H\) is Zariski connected.

Define an antiinvolution \(\sigma : G \to G\) by \(\sigma(g) := \theta(g^{-1})\).
A symmetric pair \((G, H, \theta)\) is called **good** if \(\sigma\) preserves all closed \(H \times H\) double cosets.

**Proposition**

Any connected symmetric pair over \(\mathbb{C}\) is good.

**Conjecture**

Any good symmetric pair is a Gelfand pair.

To check that a symmetric pair is Gelfand:
1. Prove that it is good
2. Prove that there are no equivariant distributions supported on the singular set in the Lie algebra \(g\).
3. Compute all the “descendants” of the pair and prove (2) for them.
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Results for $GL$ and $U$

$$(U(V \oplus W), U(V) \times U(W))$$

$$(GL(V), U(V))$$

$$(GL_n(V), U(V))$$

$$(GL_n(V), U(V))$$

$$(GL_{n+k}, GL_n \times GL_k)$$

$$(GL_n(E), GL_n(F))$$

Corollary

The pairs $(GL_n(E), GL_n(F))$ and $(GL_{n+k}, GL_n \times GL_k)$ are Gelfand pairs.
For $F = \mathbb{C}$, the pairs $(O(V \oplus W), O(V) \times O(W))$ and $(GL(V), O(V))$ are Gelfand pairs.
Let $F$ be a p-adic field. Then the following pairs are strong Gelfand pairs:

\[
(O(V \oplus F), O(V)) \downarrow \downarrow (U(V \oplus F), U(V)) \downarrow \downarrow (GL_{n+1}, GL_n)
\]
Let $F$ be a p-adic field of characteristic zero.

**Theorem (Aizenbud-Gourevitch-Rallis-Schiffmann)**

Every $GL_n(F)$-invariant distribution on $GL_{n+1}(F)$ is transposition invariant.
Let $F$ be a $p$-adic field of characteristic zero.

**Theorem (Aizenbud-Gourevitch-Rallis-Schiffmann)**

Every $GL_n(F)$-invariant distribution on $GL_{n+1}(F)$ is transposition invariant.

- $G := G_n := GL_n(F)$
- $\tilde{G} := G \rtimes \{1, \sigma\}$
- Define a character $\chi$ of $\tilde{G}$ by $\chi(G) = \{1\}$, $\chi(\tilde{G} - G) = \{-1\}$. 

Equivalent formulation:

Theorem $S^* (GL_n(F) + GL_{n+1}(F)) \tilde{G}, \chi = 0$. 

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\[ S^*(gl_{n+1}(F))^\tilde{G},\chi = 0. \]
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\[ S^* (gl_{n+1}(F)) \hat{G}, \chi = 0. \]

- \( V := F^n \)
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- \( \tilde{G} \) acts on \( X \) by
  \[
g(A, v, \phi) = (gAg^{-1}, gv, (g^*)^{-1}\phi) \\
\sigma(A, v, \phi) = (A^t, \phi^t, v^t).
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Equivalent formulation:

\[ S^*(X) \tilde{G},\chi = 0. \]

Reason:

\[ g \begin{pmatrix} A_{n\times n} & v_{n\times 1} \\ \phi_{1\times n} & \lambda \end{pmatrix} g^{-1} = \begin{pmatrix} gAg^{-1} & gv \\ (g^*)^{-1}\phi & \lambda \end{pmatrix} \]
and
\[ \begin{pmatrix} A & v \\ \phi & \lambda \end{pmatrix}^t = \begin{pmatrix} A^t & \phi^t \\ v^t & \lambda \end{pmatrix} \]
Let $\mathcal{N} \subset sl_n$ be the cone of nilpotent elements
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By Harish-Chandra descent we can assume that any $\xi \in S^*(X)\tilde{G},\chi$ is supported in $\mathcal{N} \times \Gamma$. 
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$\mathcal{N}_i := \{ a \in \mathcal{N} | \dim Ga \leq i \} \subset \mathcal{N}$
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By Harish-Chandra descent we can assume that any $\xi \in S^*(X)^{\tilde{G},\chi}$ is supported in $\mathcal{N} \times \Gamma$.

$\mathcal{N}_i := \{ a \in \mathcal{N} \mid \dim Ga \leq i \} \subset \mathcal{N}$

We prove by descending induction on $i$ that $S^*(X)^{\tilde{G},\chi} = S^*(\mathcal{N}_i \times \Gamma)^{\tilde{G},\chi}$. 
We assume $S^*(X)^{\tilde{G},\chi} = S^*(N_i \times \Gamma)^{\tilde{G},\chi}$. We want to prove that $S^*(X)^{\tilde{G},\chi} = S^*(N_{i-1} \times \Gamma)^{\tilde{G},\chi}$. 

**Reduction**
We assume $S^*(X)^{\tilde{G},\chi} = S^*(N_i \times \Gamma)^{\tilde{G},\chi}$.

We want to prove that $S^*(X)^{\tilde{G},\chi} = S^*(N_{i-1} \times \Gamma)^{\tilde{G},\chi}$.

$\nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \frac{\lambda}{n} \phi(v)Id, v, \phi)$
We assume $S^*(X)^{\tilde{G},\chi} = S^*(N_i \times \Gamma)^{\tilde{G},\chi}$. We want to prove that $S^*(X)^{\tilde{G},\chi} = S^*(N_{i-1} \times \Gamma)^{\tilde{G},\chi}$.

- $\nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \frac{\lambda}{n} \phi(v)Id, v, \phi)$

Let $\xi \in S^*(X)^{\tilde{G},\chi}$. We know that for any $\lambda$, $\xi \in S^*(\nu_\lambda(N_i \times \Gamma))^{\tilde{G},\chi}$. 


We assume $S^*(X)^\tilde{G},\chi = S^*(\mathcal{N}_i \times \Gamma)^\tilde{G},\chi$.

We want to prove that $S^*(X)^\tilde{G},\chi = S^*(\mathcal{N}_{i-1} \times \Gamma)^\tilde{G},\chi$.

Let $\xi \in S^*(X)^\tilde{G},\chi$. We know that for any $\lambda$,

$\xi \in S^*(\nu_\lambda(\mathcal{N}_i \times \Gamma))^\tilde{G},\chi$.

Let $\tilde{\mathcal{N}}_i := \bigcap_{\lambda \in F} \nu_\lambda(\mathcal{N}_i \times \Gamma)$.
We assume $S^*(X)^{\tilde{G},\chi} = S^*(N_i \times \Gamma)^{\tilde{G},\chi}$.

We want to prove that $S^*(X)^{\tilde{G},\chi} = S^*(N_{i-1} \times \Gamma)^{\tilde{G},\chi}$.

- $\nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \frac{\lambda}{n} \phi(v) \text{Id}, v, \phi)$

Let $\xi \in S^*(X)^{\tilde{G},\chi}$. We know that for any $\lambda$, $\xi \in S^*(\nu_\lambda(N_i \times \Gamma))^{\tilde{G},\chi}$.

- $\tilde{N}_i := \bigcap_{\lambda \in F} \nu_\lambda(N_i \times \Gamma)$

We know that $\xi \in S^*(\tilde{N}_i)^{\tilde{G},\chi}$. 

We assume $S^*(X)^{\tilde{G},\chi} = S^*(N_i \times \Gamma)^{\tilde{G},\chi}$.

We want to prove that $S^*(X)^{\tilde{G},\chi} = S^*(N_{i-1} \times \Gamma)^{\tilde{G},\chi}$.

\[ \nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \frac{\lambda}{n} \phi(v)Id, v, \phi) \]

Let $\xi \in S^*(X)^{\tilde{G},\chi}$. We know that for any $\lambda$, $\xi \in S^*(\nu_\lambda(N_i \times \Gamma))^{\tilde{G},\chi}$.

\[ \tilde{N}_i := \bigcap_{\lambda \in F} \nu_\lambda(N_i \times \Gamma) \]

We know that $\xi \in S^*(\tilde{N}_i)^{\tilde{G},\chi}$.

Let $O \subset N_i - N_{i-1}$ be an open orbit.
We assume $S^*(X)^{\tilde{G},\chi} = S^*(N_i \times \Gamma)^{\tilde{G},\chi}$.

We want to prove that $S^*(X)^{\tilde{G},\chi} = S^*(N_{i-1} \times \Gamma)^{\tilde{G},\chi}$.

$$\nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \frac{\lambda}{n} \phi(v)Id, v, \phi)$$

Let $\xi \in S^*(X)^{\tilde{G},\chi}$. We know that for any $\lambda$,

$\xi \in S^*(\nu_\lambda(N_i \times \Gamma))^{\tilde{G},\chi}$.

$\tilde{N}_i := \bigcap_{\lambda \in F} \nu_\lambda(N_i \times \Gamma)$

We know that $\xi \in S^*(\tilde{N}_i)^{\tilde{G},\chi}$.

Let $O \subset N_i - N_{i-1}$ be an open orbit.

$\tilde{O} := (O \times V \times V^*) \cap \tilde{N}_i$
We assume $S^*(X)^{\tilde{G},\chi} = S^*(\mathcal{N}_i \times \Gamma)^{\tilde{G},\chi}$.

We want to prove that $S^*(X)^{\tilde{G},\chi} = S^*(\mathcal{N}_{i-1} \times \Gamma)^{\tilde{G},\chi}$.

- $\nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \frac{\lambda}{n} \phi(v) Id, v, \phi)$

Let $\xi \in S^*(X)^{\tilde{G},\chi}$. We know that for any $\lambda$,

$\xi \in S^*(\nu_\lambda(\mathcal{N}_i \times \Gamma))^{\tilde{G},\chi}$.

- $\mathcal{N}_i := \bigcap_{\lambda \in F} \nu_\lambda(\mathcal{N}_i \times \Gamma)$

We know that $\xi \in S^*(\mathcal{N}_i)^{\tilde{G},\chi}$.

- Let $O \subset \mathcal{N}_i - \mathcal{N}_{i-1}$ be an open orbit.
- $\tilde{O} := (O \times V \times V^*) \cap \mathcal{N}_i$
- $\eta := \xi|_{O \times V \times V^*}$.
We assume $S^*(X)^{\tilde{G}, \chi} = S^*(\mathcal{N}_i \times \Gamma)^{\tilde{G}, \chi}$.

We want to prove that $S^*(X)^{\tilde{G}, \chi} = S^*(\mathcal{N}_{i-1} \times \Gamma)^{\tilde{G}, \chi}$.

- $\nu_{\lambda}(A, v, \phi) := (A + \lambda v \otimes \phi - \frac{\lambda}{n} \phi(v) \text{Id}, v, \phi)$

Let $\xi \in S^*(X)^{\tilde{G}, \chi}$. We know that for any $\lambda$, $\xi \in S^*(\nu_{\lambda}(\mathcal{N}_i \times \Gamma))^{\tilde{G}, \chi}$.

- $\tilde{\mathcal{N}}_i := \bigcap_{\lambda \in F} \nu_{\lambda}(\mathcal{N}_i \times \Gamma)$

We know that $\xi \in S^*(\tilde{\mathcal{N}}_i)^{\tilde{G}, \chi}$.

- Let $O \subset \mathcal{N}_i - \mathcal{N}_{i-1}$ be an open orbit.
- $\tilde{O} := (O \times V \times V^*) \cap \tilde{\mathcal{N}}_i$
- $\eta := \xi|_{O \times V \times V^*}$.

We have to show $\eta = 0$. 

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A. Aizenbud and D. Gourevitch

Invariant Distributions and Gelfand Pairs
Key Lemma

It is enough to prove

**Lemma (Key)**

Any $\eta \in S^*(O \times V \times V^*)^{\tilde{G},\chi}$ such that both $\eta$ and $\hat{\eta}$ are supported in $\tilde{O}$ is zero.
Key Lemma

It is enough to prove

**Lemma (Key)**

Any \( \eta \in S^*(O \times V \times V^*) \) such that both \( \eta \) and \( \hat{\eta} \) are supported in \( \tilde{O} \) is zero.

Apply Frobenius reciprocity:

\[
\begin{array}{ccc}
\tilde{O}_A & \rightarrow & \tilde{O} \\
\downarrow & & \downarrow \\
A & \rightarrow & O
\end{array}
\]

- \( A \in O \)
- \( \tilde{O}_A := \{ (v, \phi) \in V \times V^* \mid (A, v, \phi) \in \tilde{O} \} \)
- Let \( G_A := Stab_G(A) \) denote the centralizer of \( A \).
- \( \tilde{G}_A := Stab_{\tilde{G}}(A) \)
Equivalent formulation:

Lemma (Key’)

Any $\xi \in \mathcal{S}^* (V \times V^*) \tilde{\mathcal{G}}_{A,\chi}$ such that both $\xi$ and $\tilde{\xi}$ are supported in $\tilde{O}_A$ is zero.
Reformulation

Equivalent formulation:

Lemma (Key’)

Any $\zeta \in \mathcal{S}^*(V \times V^*) \tilde{G}_{A,\chi}$ such that both $\zeta$ and $\hat{\zeta}$ are supported in $\tilde{O}_A$ is zero.

$$Q_A := \{(v, \phi) \in V \times V^* | v \otimes \phi \in [A, gl_n]\}$$

Proposition

$\tilde{O}_A \subset Q_A$
Reformulation

Equivalent formulation:

**Lemma (Key’)**

Any \( \zeta \in S^*(V \times V^*)^{\tilde{G}_A, \chi} \) such that both \( \zeta \) and \( \hat{\zeta} \) are supported in \( \tilde{O}_A \) is zero.

\[ Q_A := \{(v, \phi) \in V \times V^* | v \otimes \phi \in [A, gl_n]\} \]

**Proposition**

\( \tilde{O}_A \subseteq Q_A \)

Now it is enough to prove

**Lemma (Key”)**

Any \( \zeta \in S^*(V \times V^*)^{\tilde{G}_A, \chi} \) such that both \( \zeta \) and \( \hat{\zeta} \) are supported in \( Q_A \) is zero.
Reduction to Jordan block

**Proposition**

\[ Q_{A \oplus B} \subset Q_A \times Q_B \]
Proposition

\[ Q_{A \oplus B} \subset Q_A \times Q_B \]

Proof.

\[
\left( \begin{array}{c} v \\ w \end{array} \right) \otimes (\phi \quad \psi) = \left( \begin{array}{ccc} v \otimes \phi & \ast \\ \ast & w \otimes \psi \end{array} \right)
\]

\[
\left[ \begin{array}{cc} A & 0 \\ 0 & B \end{array} \right], \left( \begin{array}{cc} X & Y \\ Z & W \end{array} \right) \right] = \left( \begin{array}{ccc} [A, X] & \ast \\ \ast & [B, W] \end{array} \right)
\]
**Proposition**

\[ Q_{A \oplus B} \subset Q_A \times Q_B \]

**Proof.**

\[
\begin{pmatrix} v \\ w \end{pmatrix} \otimes \begin{pmatrix} \phi & \psi \end{pmatrix} = \begin{pmatrix} v \otimes \phi & * \\ * & w \otimes \psi \end{pmatrix}
\]

\[
\begin{bmatrix} (A & 0) & (X & Y) \end{bmatrix} = \begin{bmatrix} [A, X] & * \\ * & [B, W] \end{bmatrix}
\]

Hence we can assume that \( A = J_n \) is one Jordan block.
Proof for Jordan block

\[ Q_A = \{(v, \phi) \in V \times V^* | v \otimes \phi \in [A, gl_n]\} \]
\[ Q_A = \{ (v, \phi) \in V \times V^* | v \otimes \phi \in [A, gl_n] \} = \]
\[ = \{ (v, \phi) \in V \times V^* | v \otimes \phi \perp g_A \} \]
Proof for Jordan block

\[ Q_A = \left\{ (v, \phi) \in V \times V^* \middle| v \otimes \phi \in [A, gl_n] \right\} = \]

\[ = \left\{ (v, \phi) \in V \times V^* \middle| v \otimes \phi \perp g_A \right\} = \]

\[ = \left\{ (v, \phi) \in V \times V^* \middle| \phi(Cv) = 0 \ \forall C \in g_A \right\} \]
Proof for Jordan block

\[ Q_A = \{ (v, \phi) \in V \times V^* | v \otimes \phi \in [A, gl_n] \} = \]
\[ = \{ (v, \phi) \in V \times V^* | v \otimes \phi \perp g_A \} = \]
\[ = \{ (v, \phi) \in V \times V^* | \phi(Cv) = 0 \ \forall C \in g_A \} = \]
\[ = \{ (v, \phi) \in V \times V^* | \phi(A^i v) = 0 \ \forall i \geq 0 \} \]
Proof for Jordan block

\[ Q_A = \{(v, \phi) \in V \times V^* | v \otimes \phi \in [A, gl_n]\} = \]
\[ = \{(v, \phi) \in V \times V^* | v \otimes \phi \perp g_A\} = \]
\[ = \{(v, \phi) \in V \times V^* | \phi(Cv) = 0 \ \forall C \in g_A\} = \]
\[ = \{(v, \phi) \in V \times V^* | \phi(A^i v) = 0 \ \forall i \geq 0\} \subset Z(B) \]

where \( B(v, \phi) := \phi(v) \).
Proof for Jordan block

\[ Q_A = \{(v, \phi) \in V \times V^* | v \otimes \phi \in [A, gl_n]\} = \]
\[ = \{(v, \phi) \in V \times V^* | v \otimes \phi \perp g_A\} = \]
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\[ = \{(v, \phi) \in V \times V^* | \phi(A^i v) = 0 \ \forall i \geq 0\} \subset Z(B) \]

where \( B(v, \phi) := \phi(v) \).

\( \text{Supp}(\zeta), \text{Supp}(\hat{\zeta}) \subset Z(B) \Rightarrow \zeta \) is abs-homogeneous of degree \( n \).
Denote $U := (V - \text{Ker}A^{n-1}) \times V^*$
Denote \( U := (V - \text{Ker}A^{n-1}) \times V^* \)

We have

\[ U \cap Q_A \subset V \times 0. \]
Denote \( U := (V - \text{Ker}A^{n-1}) \times V^* \)

We have 
\[ U \cap Q_A \subset V \times 0. \]

Hence \( \zeta|_U = 0. \)
Proof for Jordan block

Denote \( U := (V - \text{Ker} A^{n-1}) \times V^* \)

We have

\[ U \cap Q_A \subset V \times 0. \]

Hence \( \zeta|_U = 0 \). So \( \text{Supp}(\zeta) \subset \text{Ker} A^{n-1} \times V^* \).
Proof for Jordan block

Denote $U := (V - \text{Ker}A^{n-1}) \times V^*$

We have

$$U \cap Q_A \subset V \times 0.$$  

Hence $\zeta|_U = 0$. So $\text{Supp}(\zeta) \subset \text{Ker}A^{n-1} \times V^*$.

Similarly, $\text{Supp}(\zeta) \subset \text{Ker}A^{n-1} \times \text{Ker}(A^*)^{n-1}$.  

Denote $U := (V - \text{Ker}A^{n-1}) \times V^*$

We have

$$U \cap Q_A \subset V \times 0.$$ 

Hence $\zeta|_U = 0$. So $\text{Supp}(\zeta) \subset \text{Ker}A^{n-1} \times V^*$. Similarly, $\text{Supp}(\check{\zeta}) \subset \text{Ker}A^{n-1} \times \text{Ker}(A^*)^{n-1}$. Similarly, $\text{Supp}(\hat{\zeta}) \subset \text{Ker}A^{n-1} \times \text{Ker}(A^*)^{n-1}$. 

Hence $\zeta$ is invariant with respect to shifts by $\text{Im}A^{n-1} \times \text{Im}(A^*)^{n-1}$. Therefore $\zeta \in S^*(\text{Ker}A^{n-1}/\text{Im}A^{n-1} \times \text{Ker}(A^*)^{n-1}/\text{Im}(A^*)^{n-1}) = S^*(V^{n-2} \times V^*)$.

By induction $\zeta = 0$. □
Proving Jordan block

Denote $U := (V - \text{Ker}A^{n-1}) \times V^*$

We have

$$U \cap Q_A \subset V \times 0.$$ 

Hence $\zeta \mid_U = 0$. So $\text{Supp}(\zeta) \subset \text{Ker}A^{n-1} \times V^*$. Similarly, $\text{Supp}(\zeta) \subset \text{Ker}A^{n-1} \times \text{Ker}(A^*)^{n-1}$. Similarly, $\text{Supp}(\hat{\zeta}) \subset \text{Ker}A^{n-1} \times \text{Ker}(A^*)^{n-1}$. Hence $\zeta$ is invariant with respect to shifts by $\text{Im}A^{n-1} \times \text{Im}(A^*)^{n-1}$. 

Therefore $\zeta \in S^*(\text{Ker}A^{n-1} / \text{Im}A^{n-1} \times \text{Ker}(A^*)^{n-1} / \text{Im}(A^*)^{n-1}) = S^*(V_{n-2} \times V^*_{n-2})$.

By induction $\zeta = 0$. □
Denote $U := (V - \text{Ker}A^{n-1}) \times V^*$

We have

$$U \cap Q_A \subset V \times 0.$$  

Hence $\zeta|_U = 0$. So $\text{Supp}(\zeta) \subset \text{Ker}A^{n-1} \times V^*$.  

Similarly, $\text{Supp}(\zeta) \subset \text{Ker}A^{n-1} \times \text{Ker}(A^*)^{n-1}$.  

Similarly, $\text{Supp}(\hat{\zeta}) \subset \text{Ker}A^{n-1} \times \text{Ker}(A^*)^{n-1}$.  

Hence $\zeta$ is invariant with respect to shifts by $\text{Im}A^{n-1} \times \text{Im}(A^*)^{n-1}$. Therefore

$$\zeta \in S^*(\text{Ker}A^{n-1}/\text{Im}A^{n-1} \times \text{Ker}(A^*)^{n-1}/\text{Im}(A^*)^{n-1}) = S^*(V_{n-2} \times V^*_{n-2}).$$
Denote $U := (V - \text{Ker}A^{n-1}) \times V^*$

We have

$$U \cap Q_A \subset V \times 0.$$ 

Hence $\zeta|_U = 0$. So $\text{Supp}(\zeta) \subset \text{Ker}A^{n-1} \times V^*$. Similarly, $\text{Supp}(\zeta) \subset \text{Ker}A^{n-1} \times \text{Ker}(A^*)^{n-1}$. Similarly, $\text{Supp}(\hat{\zeta}) \subset \text{Ker}A^{n-1} \times \text{Ker}(A^*)^{n-1}$. Hence $\zeta$ is invariant with respect to shifts by $\text{Im}A^{n-1} \times \text{Im}(A^*)^{n-1}$. Therefore

$$\zeta \in S^*(\text{Ker}A^{n-1}/\text{Im}A^{n-1} \times \text{Ker}(A^*)^{n-1}/\text{Im}(A^*)^{n-1}) = S^*(V_{n-2} \times V_{n-2}^*).$$

By induction $\zeta = 0.$
Summary

Flowchart

\[ s\ell(V) \times V \times V^* \rightarrow N \times \Gamma \rightarrow N_i \times \Gamma \rightarrow \tilde{N}_i \]

\[ Q_{J_n} \leftarrow Q_A \leftarrow \tilde{O}_A \text{ Frobenius reciprocity} \text{ \leftarrow } \tilde{O} \]

\[ \text{Fourier transform and homogeneity theorem} \]

\[ Q_{J_n} + \text{ Homogeneity} \rightarrow Q_{J_{n-2}} \rightarrow \cdots \rightarrow QED \]
Let $D$ be either $F$ or a quadratic extension of $F$. Let $V$ be a vector space over $D$. Let $\langle \ , \rangle$ be a non-degenerate hermitian form on $V$. Let $W := V \oplus D$. Extend $\langle \ , \rangle$ to $W$ in the obvious way. Consider the embedding of $U(V)$ into $U(W)$. 
Let $D$ be either $F$ or a quadratic extension of $F$. Let $V$ be a vector space over $D$. Let $< , >$ be a non-degenerate hermitian form on $V$. Let $W := V \oplus D$. Extend $< , >$ to $W$ in the obvious way. Consider the embedding of $U(V)$ into $U(W)$.

**Theorem (Aizenbud-Gourevitch-Rallis-Schiffmann)**

Every $U(V)$- invariant distribution on $U(W)$ is invariant with respect to transposition.
Let $D$ be either $F$ or a quadratic extension of $F$. Let $V$ be a vector space over $D$. Let $< , >$ be a non-degenerate hermitian form on $V$. Let $W := V \oplus D$. Extend $< , >$ to $W$ in the obvious way. Consider the embedding of $U(V)$ into $U(W)$.

**Theorem**

*Every $U(V)$- invariant distribution on $U(W)$ is invariant with respect to transposition.*

- $G := U(V)$
- $\tilde{G} := G \rtimes \{1, \sigma\}$, $\chi$ as before.
- $X := su(V) \times V$
- $\tilde{G}$ acts on $X$ by $g(A, v) = (gAg^{-1}, gv)$, $\sigma(A, v) = (\overline{A}, \overline{v})$. 
Orthogonal and unitary groups

Let $D$ be either $F$ or a quadratic extension of $F$. Let $V$ be a vector space over $D$. Let $< , >$ be a non-degenerate hermitian form on $V$. Let $W := V \oplus D$. Extend $< , >$ to $W$ in the obvious way. Consider the embedding of $U(V)$ into $U(W)$.

**Theorem**

Every $U(V)$-invariant distribution on $U(W)$ is invariant with respect to transposition.

- $G := U(V)$
- $\tilde{G} := G \rtimes \{1, \sigma\}$, $\chi$ as before.
- $X := su(V) \times V$
- $\tilde{G}$ acts on $X$ by $g(A, v) = (gAg^{-1}, g v)$, $\sigma(A, v) = (\bar{A}, \bar{v})$.

Equivalent formulation:

**Theorem**

$S^*(X)\tilde{G,\chi} = 0$. 


Let $\mathcal{N} \subset su(V)$ be the cone of nilpotent elements

$\Gamma := \{ v \in V, \langle v, v \rangle = 0 \}$
Let $\mathcal{N} \subset su(V)$ be the cone of nilpotent elements

$\Gamma := \{ v \in V, \langle v, v \rangle \geq 0 \}$

By Harish-Chandra descent we can assume that any $\xi \in S^*(X)_{\tilde{G},\chi}$ is supported in $\mathcal{N} \times \Gamma$. 
Let $\mathcal{N} \subset su(V)$ be the cone of nilpotent elements
\[ \Gamma := \{ v \in V, \langle v, v \rangle = 0 \} \]

By Harish-Chandra descent we can assume that any $\xi \in S^* (X)^{\tilde{G}, \chi}$ is supported in $\mathcal{N} \times \Gamma$.

\[ \nu_\lambda (A, v) := (A + \lambda v \otimes v^t - \frac{\lambda}{n} < v, v > Id, v), \bar{\lambda} = -\lambda. \]
Sketch of the proof

- Let $\mathcal{N} \subset su(V)$ be the cone of nilpotent elements
- $\Gamma := \{ v \in V, < v, v > = 0 \}$

By Harish-Chandra descent we can assume that any $\xi \in S^\ast(X)^\widetilde{G},\chi$ is supported in $\mathcal{N} \times \Gamma$.

- $\nu_\lambda(A, v) := (A + \lambda v \otimes v^t - \frac{\lambda}{n} < v, v > Id, v), \overline{\lambda} = -\lambda$.
- $\mu_\lambda(A, v) := (A + \lambda(v \otimes v^t A + Av \otimes v^t), v)$
Let $\mathcal{N} \subset su(V)$ be the cone of nilpotent elements

$\Gamma := \{v \in V, <v, v> \geq 0\}$

By Harish-Chandra descent we can assume that any $\xi \in S^*(X)^{\tilde{G},\chi}$ is supported in $\mathcal{N} \times \Gamma$.

- $\nu_\lambda(A, v) := (A + \lambda v \otimes v^t - \frac{\lambda}{n} <v, v> Id, v)$, $\bar{\lambda} = -\lambda$.
- $\mu_\lambda(A, v) := (A + \lambda(v \otimes v^t A + Av \otimes v^t), v)$

**Lemma (Key)**

Any $\zeta \in S^*(V)^{\tilde{G}_A,\chi}$ such that both $\zeta$ and $\hat{\zeta}$ are supported in $Q_A$ is zero.