Gelfand Pairs and Invariant Distributions

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Examples

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  The same for the pair $(GL(n,\mathbb{C}), U(n))$.

Gelfand trick

Proposition (Gelfand)

Let σ be an involutive anti-automorphism of G (i.e. $\sigma(g_1 g_2) = \sigma(g_2)\sigma(g_1)$ and $\sigma^2 = Id$) and assume $\sigma(H) = H$. Suppose that $\sigma(f) = f$ for all bi H-invariant functions $f \in C(H \backslash G / H)$. Then (G, H) is a Gelfand pair.

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Sum up

Rep. theory: $\forall \rho \dim \rho^H \leq 1$

$\Longleftrightarrow$

Algebra: $C(H\backslash G/H)$ is commutative

$\Longleftarrow$

"Analysis": $\exists$ anti-involution σ s.t. $f = \sigma(f)$ $\forall f \in C(H\backslash G/H)$

$\Updownarrow$

Geometry: $\exists$ anti-involution σ that preserves H double cosets

Classical examples

Pair	Anti-involution
$(G \times G, \Delta G)$	$(g,h) \mapsto (h^{-1}, g^{-1})$
$(O(n+k), O(n) \times O(k))$	$g \mapsto g^{-1}$
$(U(n+k), U(n) \times U(k))$	
$(GL(n,\mathbb{R}), O(n))$	$g \mapsto g^t$
(G, G^θ), where G - Lie group, θ- involution, G^θ is compact	$g \mapsto \theta(g^{-1})$
(G, K), where G - is a reductive group, K - maximal compact subgroup	Cartan anti-involution

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Setting

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Definition

A local field is a locally compact non-discrete topological field.
There are 2 types of local fields of characteristic zero:

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- non-Archimedean: $\mathbb{Q}_p$ and their finite extensions

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Definition

A linear algebraic group is a subgroup of GL_n defined by polynomial equations.

Examples

GL_n, O_n, U_n, Sp_{2n},..., semisimple groups,

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Fact

Reductive groups are unimodular.

Smooth representations

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Over Archimedean F, by smooth representation V we mean a complex Fréchet representation V such that for any $v \in V$ the map $G \to V$ defined by v is smooth.

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Definition

Over non-Archimedean F, by smooth representation V we mean a complex linear representation V such that for any $v \in V$ there exists an open compact subgroup $K < G$ such that $Kv = v$.

Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M. We will consider the space $(C_c^\infty(M))^*$ of distributions on M. Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M.

Distributions

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Definition

An ℓ-space is a Hausdorff locally compact totally disconnected topological space. For an ℓ-space X we denote by $\mathcal{S}(X)$ the space of compactly supported locally constant functions on X. We let $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ be the space of distributions on X.

Gelfand Pairs

Definition

A pair of groups ($G \supset H$) is called a **Gelfand pair** if for any irreducible admissible representation ρ of G

$$dimHom_H(\rho, \mathbb{C}) \cdot dimHom_H(\widetilde{\rho}, \mathbb{C}) \leq 1$$

usually, this implies that

$$dimHom_H(\rho, \mathbb{C}) \leq 1.$$

Gelfand-Kazhdan distributional criterion

Theorem (Gelfand-Kazhdan,...)

Let σ be an involutive anti-automorphism of G and assume $\sigma(H) = H$.
Suppose that $\sigma(\xi) = \xi$ for all bi H-invariant distributions ξ on G.
Then (G, H) is a Gelfand pair.

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Corollary

Let σ be an involutive anti-automorphism of G s.t. $\sigma(H) = H$. Suppose $\sigma(\xi) = \xi$ for all distributions ξ on G invariant with respect to conjugation by H. Then (G, H) is a strong Gelfand pair.

Rep. theory: $\forall \rho \dim \rho^H \leq 1$ $\Longleftrightarrow$ Algebra: $C(H\backslash G/H)$ is commutative $\Longleftarrow$ "Analysis": $\exists\ \sigma$ s.t. $f = \sigma(f)$ $\forall f \in C(H\backslash G/H)$

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Geometry: $\exists\ \sigma$ that preserves H double cosets

Compact case

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$\dim Hom_H(\rho, \mathbb{C}) \leq 1$
$\forall \rho$

~~Algebra~~

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$\exists\ \sigma$ s.t. $\xi = \sigma(\xi)$
$\forall \xi \in \mathcal{S}^*(G)^{H \times H}$

p−adic

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Non − compact case

Results on Gelfand pairs

Pair	p-adic case	real case
$(G,(N,\psi))$	Gelfand-Kazhdan	Shalika, Kostant
$(GL_n(E),GL_n(F))$	Flicker	Aizenbud-Gourevitch
$(GL_{n+k},GL_n\times GL_k)$	Jacquet-Rallis	
$(O_{n+k},O_n\times O_k)$ over $\mathbb{C}$	————	
(GL_n,O_n) over $\mathbb{C}$		
(GL_{2n},Sp_{2n})	Heumos-Rallis	Aizenbud-Sayag
$(GL_{2n},(\begin{pmatrix} g & u \\ 0 & g\end{pmatrix},\psi))$	Jacquet-Rallis	Aizenbud-Gourevitch -Jacquet
$(GL_n,(\begin{pmatrix} SP & u \\ 0 & N\end{pmatrix},\psi))$	Offen-Sayag	Aizenbud-Offen-Sayag

- real: $\mathbb{R}$ and $\mathbb{C}$
- p-adic: $\mathbb{Q}_p$ and its finite extensions.

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Results on strong Gelfand pairs

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Remark

The results from the last two slides are used to prove splitting of periods of automorphic forms.

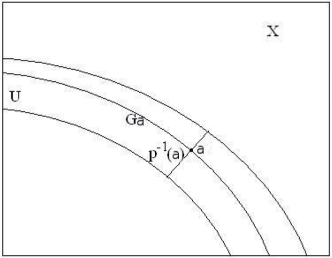
Generalized Harish-Chandra descent

Theorem

Let a reductive group G act on a smooth affine algebraic variety X. Let χ be a character of G. Suppose that for any $a \in X$ s.t. the orbit Ga is closed we have

$$\mathcal{D}(N^X_{Ga,a})^{G_a,\chi} = 0.$$

Then $\mathcal{D}(X)^{G,\chi} = 0$.



Symmetric pairs

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- We call (G, H, θ) **connected** if G/H is Zariski connected.
- Define an antiinvolution $\sigma : G \to G$ by $\sigma(g) := \theta(g^{-1})$.

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2. Prove that any H-invariant distribution on g^{σ} is σ-invariant provided that this holds outside the cone of nilpotent elements.
3. Compute all the "descendants" of the pair and prove (2) for them.

We call the property (2) regularity. We conjecture that all symmetric pairs are regular. This will imply that any good symmetric pair is a Gelfand pair.

Regular symmetric pairs

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<tr><td>$(sp_{2m}, sl_m \oplus \mathfrak{g}_a)$</td><td rowspan="7">Aizenbud</td><td rowspan="7">Sayag (based on work of Sekiguchi)</td></tr>
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<tr><td>(e_7, sl_8)</td></tr>
<tr><td>(e_8, so_{16})</td></tr>
<tr><td>$(f_4, sp_6 \oplus sl_2)$</td></tr>
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