Multiplicity One Theorems

A. Aizenbud

http://www.wisdom.weizmann.ac.il/ $\sim$ aizenr/

Formulation

Let F be a local field of characteristic zero.

Theorem (Aizenbud-Gourevitch-Rallis-Schiffmann-Sun-Zhu)

Every $GL_n(F)$-invariant distribution on $GL_{n+1}(F)$ is transposition invariant.

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It has the following corollary in representation theory.

Theorem

Let π be an irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$ and τ be an irreducible admissible representation of $\mathrm{GL}_n(F)$. Then

$$\dim Hom_{\mathrm{GL}_n(F)}(\pi, \tau) \leq 1.$$

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Similar theorems hold for orthogonal and unitary groups.

Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M. We will consider the space $(C_c^\infty(M))^*$ of distributions on M. Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M.

Distributions

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Definition

An ℓ-space is a Hausdorff locally compact totally disconnected topological space. For an ℓ-space X we denote by $\mathcal{S}(X)$ the space of compactly supported locally constant functions on X. We let $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ be the space of distributions on X.

- $\widetilde{G} := GL_n(F) \rtimes \{1, \sigma\}$
- Define a character χ of $\widetilde{G}$ by $\chi(GL_n(F)) = \{1\}$, $\chi(\widetilde{G} - GL_n(F)) = \{-1\}$.

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Proposition

Let $U \subset X$ be an open G-invariant subset and $Z := X - U$. Suppose that $\mathcal{S}^*(U)^{G,\chi} = 0$ and $\mathcal{S}^*_X(Z)^{G,\chi} = 0$. Then $\mathcal{S}^*(X)^{G,\chi} = 0$.

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$0 \to \mathcal{S}^*_X(Z)^{G,\chi} \to \mathcal{S}^*(X)^{G,\chi} \to \mathcal{S}^*(U)^{G,\chi}$. □

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For smooth manifolds, there is a slightly more complicated statement which takes into account transversal derivatives.

Frobenius descent

$$\begin{tikzcd} X_z \arrow[r] \arrow[d] & X \arrow[d] \\ z \arrow[r] & Z \end{tikzcd}$$

Theorem (Bernstein, Baruch, ...)

Let $\psi : X \to Z$ be a map.
Let G act on X and Z such that $\psi(gx) = g\psi(x)$.
Suppose that the action of G on Z is transitive.
Suppose that both G and $Stab_G(z)$ are unimodular. Then

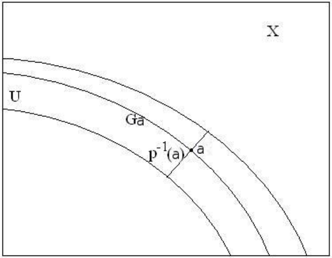
$$\mathcal{S}^*(X)^{G,\chi} \cong \mathcal{S}^*(X_z)^{Stab_G(z),\chi}.$$

Theorem

Let a reductive group G act on a smooth affine algebraic variety X. Let χ be a character of G. Suppose that for any $a \in X$ s.t. the orbit Ga is closed we have

$$\mathcal{S}^*(N_{Ga,a}^X)^{G_a,\chi} = 0.$$

Then $\mathcal{S}^*(X)^{G,\chi} = 0$.



Fourier transform

Let V be a finite dimensional vector space over F and Q be a non-degenerate quadratic form on V. Let $\widehat{\xi}$ denote the Fourier transform of ξ defined using Q.

Proposition

Let G act on V linearly and preserving Q. Let $\xi \in \mathcal{S}^*(V)^{G,\chi}$. Then $\widehat{\xi} \in \mathcal{S}^*(V)^{G,\chi}$.

Fourier transform and homogeneity

- We call a distribution $\xi \in \mathcal{S}^*(V)$ **abs-homogeneous of degree** d if for any $t \in F^\times$,
$$h_t(\xi) = u(t)|t|^d \xi,$$
where h_t denotes the homothety action on distributions and u is some unitary character of $F^\times$.

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Theorem (Jacquet, Rallis, Schiffmann,...)

*Assume F is **non-archimedean**. Let $\xi \in \mathcal{S}_V^*(Z(Q))$ be s.t. $\widehat{\xi} \in \mathcal{S}_V^*(Z(Q))$. Then ξ is abs-homogeneous of degree $\frac{1}{2}\dim V$.*

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Theorem (archimedean homogeneity)

Let F be any local field. Let $L \subset \mathcal{S}^*_V(Z(Q))$ be a non-zero linear subspace s. t. $\forall \xi \in L$ we have $\widehat{\xi} \in L$ and $Q\xi \in L$.
Then there exists a non-zero distribution $\xi \in L$ which is abs-homogeneous of degree $\frac{1}{2}dimV$ or of degree $\frac{1}{2}dimV + 1$.

Singular Support and Wave Front Set

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In the non-Archimedean case we define the singular support to be the Zariski closure of the wave front set.

Let X be a smooth algebraic variety.

- Let $\xi \in \mathcal{S}^*(X)$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))$, where $p_X : T^*X \to X$ is the projection.

Properties and the Integrability Theorem

Let X be a smooth algebraic variety.

- Let $\xi \in \mathcal{S}^*(X)$. Then $\overline{\mathrm{Supp}(\xi)}_{Zar} = p_X(SS(\xi))$, where $p_X : T^*X \to X$ is the projection.
- Let an algebraic group G act on X. Let $\xi \in \mathcal{S}^*(X)^{G,\chi}$. Then

$$SS(\xi) \subset \{(x, \phi) \in T^*X \,|\, \forall \alpha \in \mathfrak{g} \quad \phi(\alpha(x)) = 0\}.$$

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- Let V be a linear space. Let $Z \subset V^*$ be a closed subvariety, invariant with respect to homotheties. Let $\xi \in \mathcal{S}^*(V)$. Suppose that $\mathrm{Supp}(\widehat{\xi}) \subset Z$. Then $SS(\xi) \subset V \times Z$.

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- Integrability theorem:
  Let $\xi \in \mathcal{S}^*(X)$. Then $SS(\xi)$ is (weakly) coisotropic.

Coisotropic varieties

Definition

Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it ***M*-coisotropic** if the following equivalent conditions hold.

- At every smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^{\perp}$. Here, $(T_zZ)^{\perp}$ denotes the orthogonal complement to T_zZ in T_zM with respect to ω.
- The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.

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- Every non-empty coisotropic subvariety of M has dimension at least $\frac{\dim M}{2}$.

Weakly coisotropic varieties

Definition

Let X be a smooth algebraic variety. Let $Z \subset T^*X$ be an algebraic subvariety. We call it T^*X**-weakly coisotropic** if one of the following equivalent conditions holds.

- For a generic smooth point $a \in p_X(Z)$ and for a generic smooth point $y \in p_X^{-1}(a) \cap Z$ we have $CN^X_{p_X(Z),a} \subset T_y(p_X^{-1}(a) \cap Z)$.
- For any smooth point $a \in p_X(Z)$ the fiber $p_X^{-1}(a) \cap Z$ is locally invariant with respect to shifts by $CN^X_{p_X(Z),a}$.

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- Every non-empty weakly coisotropic subvariety of T^*X has dimension at least $\dim X$.

Definition

Let X be a smooth algebraic variety. Let $Z \subset X$ be a smooth subvariety and $R \subset T^*X$ be any subvariety. We define **the restriction $R|_Z \subset T^*Z$ of R to Z** by

$$R|_Z := q(p_X^{-1}(Z) \cap R),$$

where $q : p_X^{-1}(Z) \to T^*Z$ is the projection.

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Lemma

Let X be a smooth algebraic variety. Let $Z \subset X$ be a smooth subvariety. Let $R \subset T^*X$ be a (weakly) coisotropic variety. Then, under some transversality assumption, $R|_Z \subset T^*Z$ is a (weakly) coisotropic variety.

Notation

$$S := \{(A, v, \phi) \in X_n | A^n = 0 \text{ and } \phi(A^i v) = 0 \, \forall 0 \leq i \leq n\}.$$

Harish-Chandra descent and homogeneity

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By the homogeneity theorem, the stratification method and Frobenius descent we get that any $\xi \in \mathcal{S}^*(X)^{\widetilde{G},\chi}$ is supported in S'.

Reduction to the geometric statement

Notation

$$T' = \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \,|\, \forall i, j \in \{1, 2\}$$
$$(A_i, v_j, \phi_j) \in S' \text{ and } [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}.$$

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It is enough to show:

Theorem (The geometric statement)

There are no non-empty $X \times X$-weakly coisotropic subvarieties of T'.

Reduction to the Key Lemma

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Let $A \in sl(V)$ be a nilpotent Jordan block. Denote $R_A := (T' - T'')|_{\{A\} \times V \times V^*}$.

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Lemma (Key Lemma)

There are no non-empty $V \times V^* \times V \times V^*$-weakly coisotropic subvarieties of R_A.

Proof of the Key Lemma

Notation

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It is easy to see that $R_A \subset Q_A \times Q_A$ and $Q_A \times Q_A = \bigcup_{i,j=1}^{n-1} L_{ij}$, where

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$$f(v_1, \phi_1, v_2, \phi_2) := (v_1)_i(\phi_2)_{i+1} - (v_2)_i(\phi_1)_{i+1}.$$

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$$L_{ij} := (KerA^i) \times (Ker(A^*)^{n-i}) \times (KerA^j) \times (Ker(A^*)^{n-j}).$$

It is easy to see that any weakly coisotropic subvariety of $Q_A \times Q_A$ is contained in $\bigcup_{i=1}^{n-1} L_{ii}$. Hence it is enough to show that for any $0 < i < n$, we have $\dim R_A \cap L_{ii} < 2n$. Let $f \in \mathcal{O}(L_{ii})$ be the polynomial defined by

$$f(v_1, \phi_1, v_2, \phi_2) := (v_1)_i(\phi_2)_{i+1} - (v_2)_i(\phi_1)_{i+1}.$$

It is enough to show that $f(R_A \cap L_{ii}) = \{0\}$.

Proof of the Key Lemma

Let $(v_1, \phi_1, v_2, \phi_2) \in L_{ii}$. Let $M := v_1 \otimes \phi_2 - v_2 \otimes \phi_1$.

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We know that there exists a nilpotent B satisfying $[A, B] = M$. Hence this B is upper nilpotent, which implies $M_{i,i+1} = 0$ and hence $f(v_1, \phi_1, v_2, \phi_2) = 0$.

Summary

Flowchart

$$sl(V) \times V \times V^*$$

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$$sl(V) \times V \times V^* \xrightarrow[descent]{H.Ch.} S$$

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$$sl(V) \times V \times V^* \xrightarrow[\textit{descent}]{\textit{H.Ch.}} S \xrightarrow[\textit{homogeneity theorem}]{\textit{Fourier transform and}} S'$$

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$$sl(V) \times V \times V^* \xrightarrow[descent]{H.Ch.} S \xrightarrow[homogeneity\ theorem]{Fourier\ transform\ and} S' \xrightarrow[integrability\ theorem]{Fourier\ transform\ and} T'$$

Summary

Flowchart

$$\begin{tikzcd}[column sep=huge] sl(V) \times V \times V^* \arrow[r, "H.Ch."', "descent"'] & S \arrow[r, "\text{Fourier transform and}", "\text{homogeneity theorem}"'] & S' \arrow[r, "\text{Fourier transform and}", "\text{integrability theorem}"'] & T' \arrow[d] \\ & & & T' - T'' \end{tikzcd}$$

Summary

Flowchart

$$
\begin{tikzcd}
sl(V) \times V \times V^* \arrow[r, "H.Ch."', "descent"'{below}] & S \arrow[r, "\text{Fourier transform and}", "\text{homogeneity theorem}"'] & S' \arrow[r, "\text{Fourier transform and}", "\text{integrability theorem}"'] & T' \arrow[d] \\
 & & R_A & T' - T'' \arrow[l, "\text{restriction}"']
\end{tikzcd}
$$

Summary

Flowchart

$$\begin{tikzcd}
sl(V)\times V\times V^* \arrow[r, "H.Ch."', "descent"'] & S \arrow[r, "\text{Fourier transform and}", "\text{homogeneity theorem}"'] & S' \arrow[r, "\text{Fourier transform and}", "\text{integrability theorem}"'] & T' \arrow[d] \\
 & L_{ii}\cap R_A & R_A \arrow[l, "R_A\subset\bigcup L_{ij}"'] & T'-T'' \arrow[l, "restriction"']
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Summary

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sl(V)\times V\times V^* \arrow[r, "H.Ch."', "descent"'] & S \arrow[r, "\text{Fourier transform and}", "\text{homogeneity theorem}"'] & S' \arrow[r, "\text{Fourier transform and}", "\text{integrability theorem}"'] & T' \arrow[d] \\
\emptyset & L_{ii}\cap R_A \arrow[l, "f(R_A\cap L_{ii})=0"'] & R_A \arrow[l, "R_A\subset\bigcup L_{ij}"'] & T'-T'' \arrow[l, "restriction"']
\end{tikzcd}
$$