

Multiplicity One Theorems

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Theorem (Aizenbud-Gourevitch-Rallis-Schiffmann-Sun-Zhu)

Every $GL_n(F)$ -invariant distribution on $GL_{n+1}(F)$ is transposition invariant.

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Then

$$\dim \operatorname{Hom}_{GL_n(F)}(\pi, \tau) \leq 1.$$

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Similar theorems hold for orthogonal and unitary groups.

Notation

Let M be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M . We will consider the space $(C_c^\infty(M))^$ of distributions on M . Sometimes we will also consider the space $\mathcal{S}^*(M)$ of Schwartz distributions on M .*

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Definition

An ℓ -space is a Hausdorff locally compact totally disconnected topological space. For an ℓ -space X we denote by $\mathcal{S}(X)$ the space of compactly supported locally constant functions on X . We let $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ be the space of distributions on X .

- $\tilde{G} := GL_n(F) \rtimes \{1, \sigma\}$
- Define a character χ of \tilde{G} by $\chi(GL_n(F)) = \{1\}$,
 $\chi(\tilde{G} - GL_n(F)) = \{-1\}$.

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$$g \begin{pmatrix} A_{n \times n} & v_{n \times 1} \\ \phi_{1 \times n} & \lambda \end{pmatrix} g^{-1} = \begin{pmatrix} gAg^{-1} & gv \\ (g^*)^{-1}\phi & \lambda \end{pmatrix} \text{ and } \begin{pmatrix} A & v \\ \phi & \lambda \end{pmatrix}^t = \begin{pmatrix} A^t & \phi^t \\ v^t & \lambda \end{pmatrix}$$

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Let $U \subset X$ be an open G -invariant subset and $Z := X - U$. Suppose that $S^*(U)^{G,\chi} = 0$ and $S_X^*(Z)^{G,\chi} = 0$. Then $S^*(X)^{G,\chi} = 0$.

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For smooth manifolds, there is a slightly more complicated statement which takes into account transversal derivatives.

Frobenius descent

$$\begin{array}{ccc} X_Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ z & \longrightarrow & Z \end{array}$$

Theorem (Bernstein, Baruch, ...)

Let $\psi : X \rightarrow Z$ be a map.

Let G act on X and Z such that $\psi(gx) = g\psi(x)$.

Suppose that the action of G on Z is transitive.

Suppose that both G and $\text{Stab}_G(z)$ are unimodular. Then

$$S^*(X)^{G,X} \cong S^*(X_Z)^{\text{Stab}_G(z),X}.$$

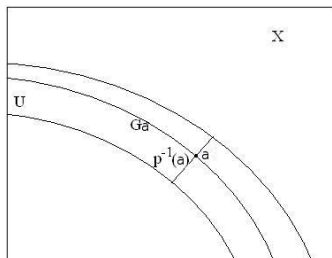
Generalized Harish-Chandra descent

Theorem

Let a reductive group G act on a smooth affine algebraic variety X . Let χ be a character of G . Suppose that for any $a \in X$ s.t. the orbit Ga is closed we have

$$S^*(N_{Ga,a}^X)^{G_a, \chi} = 0.$$

Then $S^*(X)^{G, \chi} = 0$.





Let V be a finite dimensional vector space over F and Q be a non-degenerate quadratic form on V . Let $\widehat{\xi}$ denote the Fourier transform of ξ defined using Q .

Proposition

Let G act on V linearly and preserving Q . Let $\xi \in \mathcal{S}^(V)^{G,\chi}$. Then $\widehat{\xi} \in \mathcal{S}^*(V)^{G,\chi}$.*

Fourier transform and homogeneity

- We call a distribution $\xi \in \mathcal{S}^*(V)$ **abs-homogeneous of degree d** if for any $t \in F^\times$,

$$h_t(\xi) = u(t)|t|^d \xi,$$

where h_t denotes the homothety action on distributions and u is some unitary character of F^\times .

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Theorem (Jacquet, Rallis, Schiffmann,...)

Assume F is **non-archimedean**. Let $\xi \in \mathcal{S}_V^*(Z(Q))$ be s.t. $\widehat{\xi} \in \mathcal{S}_V^*(Z(Q))$. Then ξ is abs-homogeneous of degree $\frac{1}{2} \dim V$.

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Theorem (archimedean homogeneity)

Let F be any local field. Let $L \subset \mathcal{S}_V^*(Z(Q))$ be a non-zero linear subspace s. t. $\forall \xi \in L$ we have $\widehat{\xi} \in L$ and $Q\xi \in L$.

Then there exists a non-zero distribution $\xi \in L$ which is abs-homogeneous of degree $\frac{1}{2} \dim V$ or of degree $\frac{1}{2} \dim V + 1$.

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In the non-Archimedean case we define the singular support to be the Zariski closure of the wave front set.

Properties and the Integrability Theorem

Let X be a smooth algebraic variety.

- Let $\xi \in \mathcal{S}^*(X)$. Then $\overline{\text{Supp}(\xi)}_{\text{Zar}} = p_X(\text{SS}(\xi))$, where $p_X : T^*X \rightarrow X$ is the projection.

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- Let an algebraic group G act on X . Let $\xi \in \mathcal{S}^*(X)^{G, \chi}$. Then

$$\text{SS}(\xi) \subset \{(x, \phi) \in T^*X \mid \forall \alpha \in \mathfrak{g} \quad \phi(\alpha(x)) = 0\}.$$

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- Let V be a linear space. Let $Z \subset V^*$ be a closed subvariety, invariant with respect to homotheties. Let $\xi \in \mathcal{S}^*(V)$. Suppose that $\text{Supp}(\hat{\xi}) \subset Z$. Then $SS(\xi) \subset V \times Z$.

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- Integrability theorem:
Let $\xi \in \mathcal{S}^*(X)$. Then $\text{SS}(\xi)$ is (weakly) coisotropic.

Coisotropic varieties

Definition

Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it **M -coisotropic** if the following equivalent conditions hold.

- At every smooth point $z \in Z$ we have $T_z Z \supset (T_z Z)^\perp$. Here, $(T_z Z)^\perp$ denotes the orthogonal complement to $T_z Z$ in $T_z M$ with respect to ω .
- The ideal sheaf of regular functions that vanish on \bar{Z} is closed under Poisson bracket.

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- Every non-empty coisotropic subvariety of M has dimension at least $\frac{\dim M}{2}$.

Weakly coisotropic varieties

Definition

Let X be a smooth algebraic variety. Let $Z \subset T^*X$ be an algebraic subvariety. We call it T^*X -**weakly coisotropic** if one of the following equivalent conditions holds.

- For a generic smooth point $a \in p_X(Z)$ and for a generic smooth point $y \in p_X^{-1}(a) \cap Z$ we have

$$CN_{p_X(Z), a}^X \subset T_y(p_X^{-1}(a) \cap Z).$$

- For any smooth point $a \in p_X(Z)$ the fiber $p_X^{-1}(a) \cap Z$ is locally invariant with respect to shifts by $CN_{p_X(Z), a}^X$.

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- Every non-empty weakly coisotropic subvariety of T^*X has dimension at least $\dim X$.

Definition

Let X be a smooth algebraic variety. Let $Z \subset X$ be a smooth subvariety and $R \subset T^*X$ be any subvariety. We define **the restriction** $R|_Z \subset T^*Z$ of R to Z by

$$R|_Z := q(p_X^{-1}(Z) \cap R),$$

where $q : p_X^{-1}(Z) \rightarrow T^*Z$ is the projection.

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Lemma

*Let X be a smooth algebraic variety. Let $Z \subset X$ be a smooth subvariety. Let $R \subset T^*X$ be a (weakly) coisotropic variety. Then, under some transversality assumption, $R|_Z \subset T^*Z$ is a (weakly) coisotropic variety.*

Harish-Chandra descent and homogeneity

Notation

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By the homogeneity theorem, the stratification method and Frobenius descent we get that any $\xi \in \mathcal{S}^*(X)^{\tilde{G}, X}$ is supported in S' .

Reduction to the geometric statement

Notation

$$T' = \{((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \mid \forall i, j \in \{1, 2\} \\ (A_i, v_j, \phi_j) \in S' \text{ and } [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0\}.$$

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It is enough to show:

Theorem (The geometric statement)

There are no non-empty $X \times X$ -weakly coisotropic subvarieties of T' .

Reduction to the Key Lemma

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Let $A \in sl(V)$ be a nilpotent Jordan block. Denote

$$R_A := (T' - T'')|_{\{A\} \times V \times V^*}.$$

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Lemma (Key Lemma)

There are no non-empty $V \times V^ \times V \times V^*$ -weakly coisotropic subvarieties of R_A .*

Proof of the Key Lemma

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$$Q_A := S' \cap (\{A\} \times V \times V^*) = \bigcup_{i=1}^{n-1} (\text{Ker} A^i) \times (\text{Ker}(A^*)^{n-i})$$

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$$f(v_1, \phi_1, v_2, \phi_2) := (v_1)_i(\phi_2)_{i+1} - (v_2)_i(\phi_1)_{i+1}.$$

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It is easy to see that $R_A \subset Q_A \times Q_A$ and $Q_A \times Q_A = \bigcup_{i,j=1}^{n-1} L_{ij}$, where

$$L_{ij} := (\text{Ker} A^i) \times (\text{Ker}(A^*)^{n-i}) \times (\text{Ker} A^j) \times (\text{Ker}(A^*)^{n-j}).$$

It is easy to see that any weakly coisotropic subvariety of $Q_A \times Q_A$ is contained in $\bigcup_{i,j=1}^{n-1} L_{ij}$. Hence it is enough to show that for any $0 < i < n$, we have $\dim R_A \cap L_{ii} < 2n$. Let $f \in \mathcal{O}(L_{ii})$ be the polynomial defined by

$$f(v_1, \phi_1, v_2, \phi_2) := (v_1)_i (\phi_2)_{i+1} - (v_2)_i (\phi_1)_{i+1}.$$

It is enough to show that $f(R_A \cap L_{ii}) = \{0\}$.



Proof of the Key Lemma

Let $(v_1, \phi_1, v_2, \phi_2) \in L_{ij}$. Let $M := v_1 \otimes \phi_2 - v_2 \otimes \phi_1$.

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We know that there exists a nilpotent B satisfying $[A, B] = M$. Hence this B is upper nilpotent, which implies $M_{i, i+1} = 0$ and hence $f(v_1, \phi_1, v_2, \phi_2) = 0$.

Flowchart

$$sl(V) \times V \times V^*$$

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$$sl(V) \times V \times V^* \xrightarrow[\text{descent}]{\text{H.Ch.}} \mathcal{S} \xrightarrow[\text{homogeneity theorem}]{\text{Fourier transform and}} \mathcal{S}'$$

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Flowchart

$$\begin{array}{ccccccc} sl(V) \times V \times V^* & \xrightarrow{\text{H.Ch. descent}} & S & \xrightarrow{\text{Fourier transform and homogeneity theorem}} & S' & \xrightarrow{\text{Fourier transform and integrability theorem}} & T' \\ & & & & & & \downarrow \\ & & & & & & T' - T'' \end{array}$$

Summary

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