A Quantum Analogue of Kostant's Theorem for the General Linear Group

Avraham Aizenbud

Department of Mathematics, Massachusetts Institute of Technology

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Joint with Oded Yacobi

http://math.mit.edu/~aizenr/

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Theorem (Richardson 1979)

Suppose G is semi-simple and simply connected. Then $O(G)$ is free over $O(G)^G$.

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- $gr(O(\mathfrak{g})^G) = O(\mathfrak{h})^W$.

Quantum GL_2

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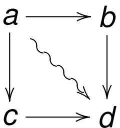

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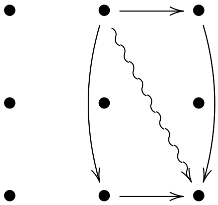

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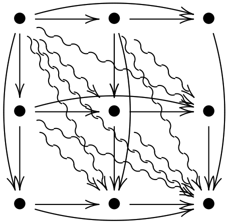

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$$I := A^{O((GL_n)_q)} = K[\Delta_1 \dots \Delta_n]$$

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$$\Delta_d = \sum_{\substack{Ind \subset \{1 \dots n\} \\ |Ind| = d}} det_q(\{x_{ij}\}_{i,j \in Ind}) \in A$$

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Suppose that q is generic and G is semi-simple algebraic group. Then the locally finite part of the quantum enveloping algebra $U_q(\mathfrak{g})$ is free over its center.

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Theorem (Baumann 2000)

Suppose that q is generic. And G is semi-simple and simply connected algebraic group. Then $O(G_q)$ is free $O(G_q)^{O(G_q)}$.

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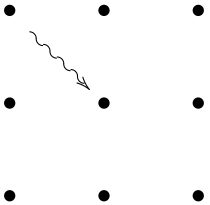

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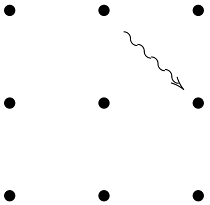

$deg([x_{12}, x_{23}]) = 0$ but $deg(x_{22}x_{13}) = 1$

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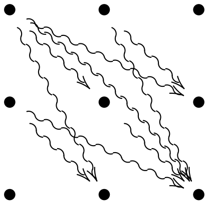

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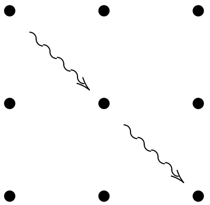

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