Counting representations of arithmetic groups and points of schemes.

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**Theorem (A.-Avni 2014)**

*Let $G$ be a semi-simple group defined over $\mathbb{Z}$ whose $\mathbb{Q}$-split rank is $> 1$.***
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**Theorem (A.-Avni 2014)**

*Let $G$ be a semi-simple group defined over $\mathbb{Z}$ whose $\mathbb{Q}$-split rank is $> 1$. Then $\zeta_G(\mathbb{Z})(40)$ converges.*
Theorem (Lubotzky-Larsen 2007)

Let $d > 2$. Any irreducible representation $\pi$ of $\text{SL}_d(\mathbb{Z})$ can be written as

$$\pi = \pi_{\text{fin}} \otimes \pi_{\text{alg}},$$

where $\pi_{\text{fin}}$ factors through $\text{SL}_d(\mathbb{Z}/N\mathbb{Z})$ and $\pi_{\text{alg}}$ extends to an algebraic representation of $\text{SL}_d(\mathbb{C})$. 

Corollary

$$\zeta_{\text{SL}_d(\mathbb{Z})}(s) = \zeta_{\text{SL}_d(\mathbb{C})}(s) = \zeta_{\text{SL}_d(\hat{\mathbb{Z}})}(s) = \prod_p \zeta_{\text{SL}_d(\mathbb{Z}_p)}(s).$$

To show that $\zeta_G(\mathbb{Z})(s)$ converges, enough to show that $\zeta_G(\mathbb{C})(s)$ converges, and $\zeta_G(\mathbb{Z}/N\mathbb{Z})(s)$ is bounded when $n$ varies.
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Corollary

To show that \( \zeta_{G(\mathbb{Z})}(s) \) converges,
Application of CSP

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To show that \( \zeta_{G(\mathbb{Z})}(s) \) converges, enough to show that \( \zeta_{G(\mathbb{C})}(s) \) converges, and \( \zeta_{G(\mathbb{Z}/N\mathbb{Z})}(s) \) is bounded when \( n \) varies.
Frobenius Formula

Theorem (Frobenius 1896)

Let $H$ be a finite group. Then

$$\zeta_H(2) = \sum_{\pi \in \text{irr}(H)} \left( \dim \pi \right)^2 = \#H$$

$$\zeta_H(0) = \sum_{\pi \in \text{irr}(H)} \left( \dim \pi \right)^0 = \#(H//H) = \#\{ (g, h) \in H^2 \mid [g, h] = 1 \}$$

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Counting representations and points
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Product of commutators of random elements

The convergence of $\zeta_G(Z)^{(40)}$ is equivalent to:

**Theorem (A.-Avni 2014)**

Let $n > 20$, and let

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\text{Def}_{n, G} = \{ (g_1, h_1, \ldots, g_n, h_n) \in G^n \mid [g_1, h_1] \cdots [g_1, h_1] = 1 \} = \text{Hom}(\pi_1(\Sigma^n), G).
$$

Then there exists a constant $C$ s.t. for any integer $k$ we have:

$$
\# \text{Def}_{n, G}(\mathbb{Z}/N\mathbb{Z}) < C \cdot \# G^{2n-1} - 1
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or equivalently:

**Theorem (A.-Avni 2014)**

For any $A \subset G(\mathbb{Z}/N\mathbb{Z})$:

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\text{Prob}([g_1, h_1] \cdots [g_n, h_n] \in A) < C \cdot \text{Prob}(g \in A),
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for random elements $g, g_1, \ldots, g_n \in G(\mathbb{Z}/N\mathbb{Z})$. 

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Theorem (Cluckers-Loser ~ 2006)

Let $X$ be an irreducible local complete intersection scheme of finite type.
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Let \(X\) be an irreducible local complete intersection scheme of finite type. Let \(n_X(p, k) = \frac{\#X(\mathbb{Z}/p^k\mathbb{Z})}{p^k \dim X}\) and \(m_X(p, k) = \frac{\#X(\mathbb{F}_p[t]/t^k)}{p^k \dim X}\). Then for almost any \(p\):

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Theorem (A.-Avni 2014)

Let $X$ be a local complete intersection, reduced, absolutely irreducible scheme of finite type over $\mathbb{Z}$, s.t. $X_{\mathbb{Q}}$ has rational singularities. Then:

$$\text{abscissa of convergence of } \sum_{N=1}^{\infty} \frac{|X(\mathbb{Z}/N\mathbb{Z})| \cdot N^{-s}}{s} = \dim X_{\mathbb{Q}} + 1.$$

The function $P_X(s)$ can be analytically continued to $\{s | \Re(s) > \dim X_{\mathbb{Q}} + 1/2\}$.

The only pole of the continued function on the line $\Re(s) = \dim X_{\mathbb{Q}} + 1$ is a simple pole at $\dim X_{\mathbb{Q}} + 1$. 
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Theorem (A.-Avni 2013)

Let $n > 20$. Then the singularities of the deformation variety $\text{Def}_{G,n}$ are rational (and complete intersection).
Rationality of the singularities of moduli spaces

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**Corollary (A.-Avni 2013)**

The moduli spaces of $G$ local systems on a genus $n$ surface have rational singularities.
\{(x, y, z)|z^2 = x^2 + y^2\} \text{ have rational singularities}

\[\Downarrow\]
\[\text{def}_{g,n} := \{(g_1, h_1, \ldots g_n, h_n) \in g^{2n}|[g_1, h_1] + \cdots + [g_1, h_1] = 0\}\]
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\[\Downarrow\]
\[\text{Def}_{G,n} \text{ have rational singularities at 1}\]

\[\Updownarrow\]
\[\exists m \text{ s.t.}\ \#\{(g_1, h_1, \ldots g_n, h_n) \in G(\mathbb{Z}/p^k\mathbb{Z})^{2n}| [g_1, h_1] \cdots [g_n, h_n] = 1; g_i = h_i = 1 \mod p^m\} = p^{(2n-1)(k-m)\dim G}(1 + O(p^{-\frac{1}{2}}))\]

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*We have the following implications:*

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\zeta_{G}(\mathbb{Z}) (2n - 2) < \infty.
$$

$$
\Downarrow
$$

$$
\zeta_{G}(\mathbb{Z}_p) (2n - 2) < \infty \text{ for any } p.
$$

$$
\Uparrow
$$

**Def** $G$, $n$ has rational singularities.

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All of the above happens for $n > 20$. 
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Pushforward of smooth measures

Let \( \Phi_{G, n} : G_2^n \to G \) be defined by:

\[
\Phi_{G, n}(g_1, h_1, \ldots, g_n, h_n) := [g_1, h_1] \cdots [g_n, h_n].
\]

Let \( \mu \) be the Haar measure on \( G(\mathbb{Z}_p) \).

The convergence of \( \zeta_{G(\mathbb{Z}_p)}(2^n - 2) \) is equivalent to the fact that \( \Phi(\mu) = f \cdot \mu \) for a continuous function \( f \).

Theorem (A.-Avni, 2013)

Let:

\[ m_{X, \phi} \to Y \]

s.t. \( \phi \) is a flat morphism of smooth algebraic varieties over a local field \( F \), s.t. all its fibers are of rational singularities (in what follows: FRS morphism).

\( m \) is a Schwartz (i.e. compactly supported locally Haar) measure on \( X(\mathbb{F}) \).

Then \( \phi^* m \) has continuous density.
Let $\Phi_{G,n} : G^{2n} \to G$ be defined by:

$$\Phi_{G,n}(g_1, h_1, \ldots, g_n, h_n) := [g_1, h_1] \cdots [g_n, h_n].$$
Let $\Phi_{G,n} : G^{2n} \to G$ be defined by:

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Let $\mu$ be the Haar measure on $G(\mathbb{Z}_p)$. 

Theorem (A.-Avni, 2013) 

Let:

- $m$ be a Schwartz (i.e. compactly supported locally Haar) measure on $X(F)$.
- $\phi : X \to Y$ be a flat morphism of smooth algebraic varieties over a local field $F$, s.t. all its fibers are of rational singularities (in what follows: FRS morphism).

Then $\phi^* (m)$ has continuous density.
Pushforward of smooth measures

Let $\Phi_{G,n} : G^{2n} \to G$ be defined by:

$$\Phi_{G,n}(g_1, h_1, \ldots, g_n, h_n) := [g_1, h_1] \cdots [g_n, h_n].$$

Let $\mu$ be the Haar measure on $G(\mathbb{Z}_p)$. The convergence of $\zeta_{G(\mathbb{Z}_p)}(2n-2)$ is equivalent to the fact that $\Phi(\mu) = f \cdot \mu$ for a continuous function $f$. 

Theorem (A.-Avni, 2013)

Let:

$m : X \to Y$ s.t. $\phi$ is a flat morphism of smooth algebraic varieties over a local field $F$, s.t. all its fibers are of rational singularities (in what follows: FRS morphism).

$m$ is a Schwartz (i.e. compactly supported locally Haar) measure on $X(F)$.

Then $\phi_*(m)$ has continuous density.
Let $\Phi_{G,n} : G^{2n} \to G$ be defined by:

$$\Phi_{G,n}(g_1, h_1, \ldots, g_n, h_n) := [g_1, h_1] \cdots [g_n, h_n].$$

Let $\mu$ be the Haar measure on $G(\mathbb{Z}_p)$. The convergence of $\zeta_{G(\mathbb{Z}_p)}(2n - 2)$ is equivalent to the fact that $\Phi(\mu) = f \cdot \mu$ for a continuous function $f$.

**Theorem (A.-Avni, 2013)**

Let:

$$m \quad X \overset{\phi}{\to} Y$$

s.t.
Let $\Phi_{G,n} : G^{2n} \to G$ be defined by:

$$\Phi_{G,n}(g_1, h_1, \ldots, g_n, h_n) := [g_1, h_1] \cdots [g_n, h_n].$$

Let $\mu$ be the Haar measure on $G(\mathbb{Z}_p)$. The convergence of $\zeta_{G(\mathbb{Z}_p)}(2n - 2)$ is equivalent to the fact that $\Phi(\mu) = f \cdot \mu$ for a continuous function $f$.

**Theorem (A.-Avni, 2013)**

Let:

$$\begin{array}{ccc}
\mu & \phi & Y \\
X & \phi & Y
\end{array}$$

s.t.

- $\phi$ is a flat morphism of smooth algebraic varieties over a local field $F$, s.t. all its fibers are of rational singularities.
Let $\Phi_{G,n} : G^{2n} \to G$ be defined by:

$$\Phi_{G,n}(g_1, h_1, \ldots, g_n, h_n) := [g_1, h_1] \cdots [g_n, h_n].$$

Let $\mu$ be the Haar measure on $G(\mathbb{Z}_p)$. The convergence of $\zeta_{G(\mathbb{Z}_p)}(2n - 2)$ is equivalent to the fact that $\Phi(\mu) = f \cdot \mu$ for a continuous function $f$.

**Theorem (A.-Avni, 2013)**

Let:

$$\begin{array}{ccc}
X & \xrightarrow{\phi} & Y \\
\downarrow{m} & & \\
X & \phi & Y
\end{array}$$

s.t.

- $\phi$ is a flat morphism of smooth algebraic varieties over a local field $F$, s.t. all its fibers are of rational singularities (in what follows: FRS morphism).
Pushforward of smooth measures

Let $\Phi_{G,n} : G^{2n} \to G$ be defined by:

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**Theorem (A.-Avni, 2013)**

Let:

$$m \overset{\phi}{\to} Y$$

s.t.

- $\phi$ is a flat morphism of smooth algebraic varieties over a local field $F$, s.t. all its fibers are of rational singularities (in what follows: FRS morphism).
- $m$ is a Schwartz (i.e. compactly supported locally Haar) measure on $X(F)$. 
Let $\Phi_{G,n} : G^{2n} \to G$ be defined by:

$$\Phi_{G,n}(g_1, h_1, \ldots, g_n, h_n) := [g_1, h_1] \cdots [g_n, h_n].$$

Let $\mu$ be the Haar measure on $G(\mathbb{Z}_p)$. The convergence of $\zeta_{G(\mathbb{Z}_p)}(2n - 2)$ is equivalent to the fact that $\Phi(\mu) = f \cdot \mu$ for a continuous function $f$.

**Theorem (A.-Avni, 2013)**

Let:

$$\begin{align*}
X & \overset{\phi}{\to} Y \\
m & \in \text{X}(F)
\end{align*}$$

s.t.

- $\phi$ is a flat morphism of smooth algebraic varieties over a local field $F$, s.t. all its fibers are of rational singularities (in what follows: FRS morphism).
- $m$ is a Schwartz (i.e. compactly supported locally Haar) measure on $X(F)$.

Then $\phi_*(m)$ has continuous density.
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}
$A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$
A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \epsilon \}
$A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon\}$

![Graph showing the region $A$ and its area $\text{Area}(A)/\varepsilon$ as a function of $\varepsilon$.](image)
A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}

\frac{\text{Area}(A)}{\varepsilon}

\begin{align*}
\text{Area}(A)/\varepsilon & \quad 0 & 0.02 & 0.04 & 0.06 & 0.08 \\
0 & 5 & 10 & 15 & 20 & 25
\end{align*}
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \epsilon \} \]
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
$A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$
$A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$
A = \{x,y \text{ s.t. } |x^2+y^2|<1 \text{ and } |x^2-y^2|<\varepsilon \}
A = \{x,y \text{ s.t. } |x^2+y^2|<1 \text{ and } |x^2-y^2|<\varepsilon\}
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
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A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \epsilon \}
$A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}\}
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
\( A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \)
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
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$A = \{ x,y \text{ s.t. } |x^2+y^2|<1 \text{ and } |x^2-y^2|<\varepsilon \}$
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
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$A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$
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$A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$
$A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$

\[ \frac{\text{Area}(A)}{\varepsilon} \]

![Graph showing the area of set $A$ as a function of $\varepsilon$.](image_url)
$A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$

The graph shows a region defined by the given inequality, along with a plot of the ratio of the area $A$ to $\varepsilon$ against $\varepsilon$. The area decreases as $\varepsilon$ increases, approaching zero as $\varepsilon$ approaches infinity.
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}
$A = \{ x, y : |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \epsilon \}$
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
$A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$

Area(A)/\varepsilon

0 0.02 0.04 0.06 0.08

0 5 10 15 20 25

$\varepsilon$
$A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$

\[
\frac{\text{Area}(A)}{\varepsilon}
\]
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \epsilon \} \]
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$A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$

$\frac{\text{Area}(A)}{\varepsilon}$

Graph showing the area $A$ as a function of $\varepsilon$.
\[ A = \{x,y \text{ s.t. } |x^2+y^2|<1 \text{ and } |x^2-y^2|<\varepsilon \} \]
A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon\}
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
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\[
\frac{\text{Area}(A)}{\varepsilon}
\]
$A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$

Area(A)/\varepsilon

0 0.02 0.04 0.06 0.08

0 5 10 15 20 25

$\varepsilon$
$A = \{x,y \text{ s.t. } |x^2+y^2|<1 \text{ and } |x^2-y^2|<\varepsilon \}$
A = \{x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}
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\frac{\text{Area}(A)}{\varepsilon}
\[ A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \} \]
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$A = \{ x, y \text{ s.t. } |x^2 + y^2| < 1 \text{ and } |x^2 - y^2| < \varepsilon \}$
\[ V = \{ x, y, z \text{ s.t. } |x^2+y^2+z^2| < 1 \text{ and } |x^2+y^2-z^2| < \varepsilon \} \]
$V = \{ x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon \}$
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$V = \{ x, y, z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\epsilon \}$
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$V = \{x,y,z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon\}$
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V = \{x, y, z \text{ s.t.} |x^2 + y^2 + z^2| < 1 \text{ and } |x^2 + y^2 - z^2| < \varepsilon \}
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\[ V = \{ x, y, z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon \} \]
\[ V = \{ x, y, z \text{ s.t. } |x^2 + y^2 + z^2| < 1 \text{ and } |x^2 + y^2 - z^2| < \varepsilon \} \]

\[ \frac{Vol(V)}{\varepsilon} \]

\begin{tabular}{c c c c c}
0.02 & 0.04 & 0.06 & 0.08 & 0.10 \\
0 & 5 & 10 & 15 & 20 \\
\end{tabular}
V = {x,y,z s.t. |x^2+y^2+z^2| < 1 and |x^2+y^2-z^2| < \varepsilon}
$V = \{ x, y, z \text{ s.t. } |x^2 + y^2 + z^2| < 1 \text{ and } |x^2 + y^2 - z^2| < \epsilon \}$
$V = \{ x, y, z \text{ s.t. } |x^2 + y^2 + z^2| < 1 \text{ and } |x^2 + y^2 - z^2| < \varepsilon \}$
\[ V = \{ x, y, z \text{ s.t. } |x^2 + y^2 + z^2| < 1 \text{ and } |x^2 + y^2 - z^2| < \varepsilon \} \]
\[ V = \{ x, y, z \text{ s.t. } |x^2 + y^2 + z^2| < 1 \text{ and } |x^2 + y^2 - z^2| < \varepsilon \} \]
\[ V = \{x, y, z \text{ s.t. } |x^2+y^2+z^2|<1 \text{ and } |x^2+y^2-z^2|<\varepsilon \} \]
\[ V = \{ x,y,z \text{ s.t. } |x^2+y^2+z^2| < 1 \text{ and } |x^2+y^2-z^2| < \varepsilon \} \]
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$\frac{\text{Vol}(V)}{\epsilon}$ vs $\epsilon$
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Jet schemes and rational singularities

Definition

For a scheme $X$ defined over $k$, the jet scheme $\text{jet}_n(X)$ is the natural scheme defined over $k$ s.t. $X(k[t]/t^n) \sim \text{jet}_n(X)(k)$.

Theorem (Mustata 2001)

Assume that $X$ is a local complete intersection connected variety. TFAE:

1. $X$ is irreducible and has rational singularities.
2. The jet schemes of $X$ are irreducible.
Jet schemes and rational singularities

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Theorem (Mustata 2001)

Assume that $X$ is a local complete intersection connected variety. TFAE:

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- The jet schemes of $X$ are irreducible.
\[ \lim_{p \to \infty} m_X(p, k) = 1 \]

\[ \sup_{k} n_X(p, k) < \infty \]

\[ \lim_{q \to \infty} \sup_{k} h_X(q, k) = 1 \]

X has rat. sing.

Continuity Criterion

Mustata, L-W, Chebotarev

Continuity Criterion

Mustata Thm., L-W Bounds

Obvious

Chebotarev thm.,
L-W Bounds,
Motivic int.

Obvious

Obvious