Derivatives for representations of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$

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The p-adic case

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Definition

$$P_n = \left\{ \begin{pmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subset G_n := GL_n$$

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Proof.

$$\mathcal{M}(P_n) = \mathcal{M}(\mathcal{H}(P_n)) = \mathcal{M}(\mathcal{H}(G_{n-1} \ltimes \mathbb{A}^{n-1})) =$$
$$= \mathcal{M}(\mathcal{H}(G_{n-1}) \otimes \mathcal{H}(\mathbb{A}^{n-1})) \cong \mathcal{M}(\mathcal{H}(G_{n-1}) \otimes \mathcal{S}(\mathbb{A}^{n-1}))$$

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- $D^k = \Psi \circ \Phi^{k-1}$

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Theorem (Casselman-Wallach)

The functor $HC : \mathcal{M}_\infty(G) \to \mathcal{M}_{HC}(G)$ is an equivalence of categories.

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- $D_3^k(\pi) = (D_1^k(\pi))_{\mathfrak{v}_{n-k+1}}$.
- $depth(\pi)$ – the largest part in the associated partition of π

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- $D_i^n = (\Phi)^{n-1}$ is the Whittaker functor.

$$depth(\pi) = n \iff D_2^n(\pi) \neq 0$$

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6. Using the exactness we prove the product formula in the smooth category
7. We deduce from the product formula that for a unitarizable representation π

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$$\Delta_{4m-4} \twoheadleftarrow \chi_1|_{G_{m-1}} \times \chi_2|_{G_{m-1}} \times \chi_3|_{G_{m-1}} \times \chi_4|_{G_{m-1}} =$$
$$= D_1^4(\chi_1 \times \chi_2 \times \chi_3 \times \chi_4) \twoheadrightarrow D_1^4(\Delta_{4m}) \twoheadrightarrow A(\Delta_{4m})$$

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- Associated variety – $AV(\pi)$

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We know – $D_1^d(\pi)$ is finitely generated over $\mathfrak{n}_{n-d+1}$
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$depth(\pi) = d \Rightarrow$ constrains on $\mathcal{V}_{\mathfrak{g}}(\pi) \Rightarrow$

$\Rightarrow AV_{\mathfrak{n}_{n-d+1}}(D_1^d(\pi)) \subset \mathfrak{n}_{n-d}^* \Rightarrow D_1^d(\pi)$ is f.g. over $\mathfrak{n}_{n-d}$

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Solution – to introduce a class of “good" $\mathfrak{p}_n$ representations

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- $\Phi(\mathcal{S}(P_n/Q)) = \mathcal{S}(Z_0)$ *for suitable* $Z_0 \subset Z := P_n/(QV_n)$

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Method – exactness, key lemma, induction