# Smooth Transfer of Kloosterman Integrals

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January 27, 2011

Joint with Dmitry Gourevitch

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# Motivation

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Study of $End(\mathcal{S}(G/H))$

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$$\begin{tikzcd}[row sep=large]
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\text{Harmonic analysis on } G \text{ w.r.t. the two-sided action of } G \times G \\
\text{Harmonic analysis on } G/H \text{ w.r.t. the action of } G \arrow[u, Rightarrow] \arrow[d, Leftrightarrow] \\
\text{Study of } End(\mathcal{S}(G/H)) \\
\text{Study of } Hom(\mathcal{S}(G/H_1), \mathcal{S}(G/H_2)) \arrow[u, Rightarrow]
\end{tikzcd}$$

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Study of $\mathcal{S}(X)_G$

$\Updownarrow$ (in many cases)

Study of the image of $\Omega_G : \mathcal{S}(X) \to F(X//G)$

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Local smooth transfer & Fundamental Lemma

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**Situation 2**

*$U(E)$ acts on $H$, where*

- *$H$ is the space of non-degenerate hermitian matrices of size $n$.*
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**Theorem**

$$\widetilde{\Omega}(\mathcal{S}(M)) = \widetilde{\Omega}^E(\mathcal{S}(M^E))$$

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**Proposition**

$$\begin{tikzcd} \mathcal{S}(O_i) \arrow[r, "\Omega_i"] \arrow[rr, bend left, "\Omega"] & \mathcal{S}(GL_i \times M_{n-i}) \arrow[r, "\Omega"] & C^\infty(T) \end{tikzcd}$$

# Fourier transform, Jacquet transform and the inversion formula

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**Theorem (Jacquet)**

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\mathcal{S}(M) \arrow[r, "\mathcal{F}"] \arrow[d, "\widetilde{\Omega}"'] & \mathcal{S}(M) \arrow[d, "\widetilde{\Omega}"] \\
C^\infty(T) \arrow[r, "\mathfrak{J}"] & C^\infty(T)
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**Proof.**

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4. Key Lemma $\not\Leftrightarrow$ co-Key Lemma

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    - $gr_k(\mathcal{S}^*_X(Z)) = \mathcal{S}^*(Z, Sym^k(CN_Z^X))$
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**Example**

$$\mathcal{S}(\mathbb{R})/\mathcal{S}(\mathbb{R}^\times) = \mathbb{C}[[t]]$$

**Lemma**

*Let* $W \subset V$ *linear spaces s.t.* $W \not\supset W^{\perp}$ *. Then*

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We need dual of the following lemma

## Theorem

*Let* $G$ *act on* $X, E$ *and* $Z \subset X$ *closed subvariety s.t.*

$$\forall z \in Z, k \in \mathbb{Z}_{\geq 0} : (E|_z^* \otimes Sym^k(N_{z,Gz}^X) \otimes \Delta_{G,G_z})^{G_z} = 0.$$

*Then*

$$\mathcal{S}_X^*(Z, E)^G = 0.$$

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**Example**

$X = Z$ and we have a submersion $X \to W$ with all fibers - orbits.