Fourier Transform of Algebraic Measures

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Then the Fourier transform of $|p|$ is smooth in an open dense set U. Moreover, U is explicitly described by p in algebro-geometric terms.

The Archimedean Case

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Theorem (Malgrange, Kashiwara-Kawai-Sato, Gabber)

$SS(M)$ is coisotropic.

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In fact we do not require that X be closed, but rather we ask for a certain regular behaviour of ω near the boundary of X which will allow us to consider $|\omega|$ as a distribution on W.

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- Let V be a linear space. Let $Z \subset V^*$ be a closed subvariety, invariant with respect to homotheties. Let $\xi \in \mathcal{S}^*(V)$. Suppose that $WF(\widehat{\xi}) \subset Z \times V$. Then $WF(\xi) \subset V \times Z$.

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- Behavior w.r.t. group action:
  Let an algebraic group G act on X. Let $\xi \in \mathcal{S}^*(X)^{G,\chi}$. Then

$$WF(\xi) \subset \{(x, \phi) \in T^*X \,|\, \forall \alpha \in \mathfrak{g}, \quad \phi(\alpha(x)) = 0\} = \bigcup_{y \in X} CN_{Gy}^X.$$

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  Let $p : X \to Y$ be a proper analytic submersion. Then
  $WF(p_*(\xi)) \subset p_*(WF(\xi))$.

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- Behavior w.r.t. direct image:
  Let $p : X \to Y$ be a proper analytic submersion. Then $WF(p_*(\xi)) \subset p_*(WF(\xi))$.
- Weak integrability theorem:
  Let $\xi \in \mathcal{S}^*(X)$. Then $WF(\xi)$ is weakly coisotropic.

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Let Y be smooth algebraic variety and $X \subset Y \times W$. Let ω be an algebraic top differential form on it. Then the partial Fourier transform of $|\omega|$ w.r.t. W is WF-holonomic.

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Let X be an algebraic variety. Then there exists a resolution of singularities $p : Y \to X$, i.e. a proper surjective map which is isomorphism on an open dense set.
Moreover, let $D \subset X$ be a divisor. Then we can take p s.t. $p^{-1}(D)$ will be a divisor with normal crossings.

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- $i : U \to Graph(f) \to Y \times \mathbb{A}^n$.

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Assume that $Z_1 \cup Z_2$ *is a divisor with normal crossings. Then the partial Fourier transform of* $i_* \left|(\omega|_U)\right|$ *w.r.t.* $\mathbb{A}^n$ *is WF-holonomic.*

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Let Y be smooth analytic variety and f be a meromorphic function on Y. Let $Z_1 := f^{-1}(\infty)$ and let $U = Y - Z_1$. Let $i : U \to Graph(f) \to Y \times F$.

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Local model:

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Local model: f and ω are monomials.