WF-holonomicity of constructible distributions on non-Archimedean local fields

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Distributions

Definition

Test functions – smooth compactly supported functions / Schwartz functions.

Distributions – functionals on test functions.

Examples

Delta function \( \delta \), its derivative \( \delta' \), any locally \( L^1 \) function, \( S^p_S^\lambda \).

Operations with distributions:

- pullback
- push forward
- Fourier transform
- Derivation

Algebraic operations:
Distributions

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Examples
δ, its derivative δ', any locally L¹ function, etc.

Operations with distributions:
pullback
push forward
Fourier transform
Derivation
Algebraic operations:

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Distributions

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Examples

- Delta function $\delta$
- Derivative of the delta function $\delta'$
- Any locally $L^1$ function
- $S^p_S$$^\lambda$

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**Operations with distributions:**
- pullback
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- Algebraic operations: $+, \cdot, \boxdot$
The Archimedean case

**Definition**

Holonomic distributions – distributions that satisfy lots of PDE:

Let $\xi > S^\hat{\cdot}$ be a distribution on vector space. $\xi$ is holonomic iff

$$\dim \text{Char} \hat{\xi} = \dim \text{Zeros} \hat{\xi} \text{Sym} \hat{\xi} \leq \dim V.$$  

**Theorem (Bernstein 1970)**

The class of holonomic distributions is closed under all of the operations above whenever these are defined.

$$\dim \text{Char} \hat{\xi} \leq \dim V.$$  

**Theorem (Kashiwara-Kawai-Sato, Malgrange, Gaber 1980)**

$\text{Char} \hat{\xi}$ is co-isotropic.

"All the distributions which appear in nature are holonomic." - A. Aizenbud
The Archimedean case

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“All the distributions which appear in nature are holonomic.”
Wave front set

Observation \( \hat{\xi} \) is smooth iff \( \hat{\xi} \) is rapidly decaying.

Definition

Let \( \xi > S \hat{\xi} \). \( \hat{\xi} \) is a distribution on vector space. We say that \( \xi \) is smooth at point \( x \) and direction \( v \) if \( \hat{\rho \xi} \) is rapidly decaying at direction \( v \), where \( \rho \) is a cut-off function of a small enough neighborhood of \( x \).

\[ \text{WF} \hat{\xi} \]

Theorem (Hörmander 1980) \( \text{WF} \hat{\xi} \) is invariant w.r.t. diffeomorphisms.

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Let \( \xi \in S^* (V) \) is a distribution on vector space.

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- \( WF(\xi) = \{(x, v) \in T^*V | \xi \text{ is not smooth at } (x, v)\} \).
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- \( \text{WF}(\xi) \) is invariant w.r.t. diffeomorphisms.
- \( \text{WF}(\xi) \subset \text{Char}(\xi) \).
**p-adic numbers**

**Definition**

P-adic numbers are "numbers" who have a "p-cimal" presentation which is finite after the "p-cimal point" and possibly infinite before it.

Alternatively: The field of p-adic numbers \( \mathbb{Q}_p \) is the completion of \( \mathbb{Q} \) w.r.t. the p-adic norm:

\[
V_p = \frac{\mathbb{Z}}{p^{\mathbb{N}}}.
\]

Although we consider p-adic numbers as arguments, the values of our functions are always complex. Smooth functions on \( \mathbb{Q}_p \) are locally constant functions. Rapidly decaying functions are functions with compact support. This gives us the notion of distribution.

Instead of using the periodic exponent \( e^{ix} \) one uses a fixed additive character \( \psi^* \). This gives us the notion of Fourier transform and wave front set.

/ No action of differential operators on distributions.
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\left| p^k \frac{m}{n} \right| = p^{-k}, \quad \text{where: } \gcd(p, n) = \gcd(p, m) = 1.
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Wave front holonomicity

Theorem (A. 2008)

\[ \text{WF} \hat{\xi} \] includes Lagrangian, in particular
\[ \dim \text{WF} \hat{\xi} \leq \dim V. \]

Definition
\( \xi \) is WF-holonomic if \( \text{WF} \hat{\xi} \) is isotropic. In particular
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Theorem (A.-Drinfeld 2011)

Many distributions with algebraic description (and their Fourier transforms) are WF-holonomic.
WF-holonomicity is stable under proper push-forward and submersive pull-back.

/ WF-holonomicity is not stable under Fourier transform.
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Constructible functions

Functions that have a nice formula.

Examples
- Absolute value of a rational function.
- Valuation (log of the absolute value) of a rational function.
- $\psi$ composed with a rational function.
- Characteristic function of a ball.

Definition
The algebra of constructible functions is the algebra generated by (generalizations of) the above examples.

Non-example:
$\log$.  

Theorem (Clukers-Loeser 2005)
The class of constructible functions is closed under the above operations, whenever defined.

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“All the functions which appear in nature are constructible.”
(p-adic) Wavelet transform

Definition

Let $F$ be a $p$-adic (more generally non-Archimedean local) field. Define:

$$
\text{WL} S \hat{\xi} \hat{V} C \hat{a}, b \hat{\xi, 1} B \hat{a}, S \hat{e}
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It is easy to see that $\text{WL}$ is 1-1.
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$$WL : S^*(V) \rightarrow C^\infty(V \times F^\times)$$

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WL(\xi)(a, b) := \langle \xi, 1_{B(a, |b|)} \rangle
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$$WL : S^*(V) \to C^\infty(V \times F^\times)$$

$$WL(\xi)(a, b) := \langle \xi, 1_{B(a, |b|)} \rangle$$

It is easy to see that $WL$ is 1-1.
Constructible distributions

Definition

$\xi$ is constructible iff $\hat{\mathcal{L}} \xi$ is constructible.

Theorem (Cluckers-Halupczok-Loeser-Raibaut, 2018)

The class of constructible distributions is closed under all the above operations, whenever defined.

Constructible distributions are smooth almost everywhere.

"All the distributions which appear in nature are constructible"

A. Aizenbud
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\( \xi \) is constructible iff \( WL(\xi) \) is constructible.

**Theorem (Cluckers-Halupczok-Loeser-Raibaut, 2018)**

The class of constructible distributions is closed under all the above operations, whenever defined.
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**Theorem (Cluckers-Halupczok-Loeser-Raibaut, 2018)**

- The class of constructible distributions is closed under all the above operations, whenever defined.
- Constructible distributions are smooth almost everywhere.
**Constructible distributions**

**Definition**

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**Theorem (Cluckers-Halupczok-Loeser-Raibaut, 2018)**

- The class of constructible distributions is closed under all the above operations, whenever defined.
- Constructible distributions are smooth almost everywhere.

“All the distributions which appear in nature are constructible”
Main Result

Theorem (A.-Cluckers 2019)
Constructible distributions are WF-holonomic.

Main ingredients of the proof.

Regularization: a constructible distribution can be extended from an open set.

Resolution of singularities in the constructible (in fact, definable) setting.

Key-Lemma: a smooth constructible function on an open set can be extended to a holonomic constructible distribution.
Theorem (A.-Cluckers 2019)

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*Constructible distributions are WF-holonomic.*

Main ingredients of the proof.

- **Regularization:** a constructible distribution can be extended from an open set.
- **Resolution of singularities in the constructible (in fact, definable) setting.**
- **Key-Lemma:** a smooth constructible function on an open set can be extended to an holonomic constructible distribution.
Idea of the proof.

ξ

S
U

is smooth for open dense
U.

Extend ξ
S
U

to an holonomic constructible distribution
ξ
œ

Let
η
ξ
œ
ξ

We have

dim
supp
ˆ
η
µ

½

V

Resolve
Z

supp
ˆ
η
µ

by a smooth manifold:
P
M
Z

Let
Z
œ
`Z
open dense s.t.
p
1
ˆ
Z
œ
µ
Z
œ

Extend
p
‡
ˆ
η
S
Z
œ
µ

on
M

By the induction assumption,
µ
is WF-holonomic.

Thus
p
‡
ˆ
µ
µ
is constructible WF-holonomic.

By the induction assumption
p
‡
ˆ
µ
µ
η
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Idea of the proof.

- $\xi|_U$ is smooth for open dense $U$. 
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- $\xi|_U$ is smooth for open dense $U$.
- Extend $\xi|_U$ to an holonomic constructible distribution $\xi'$.

Let $\eta = \xi'$. We have $\dim \operatorname{supp} \hat{\eta} = \dim V$.

Resolve $Z = \operatorname{supp} \hat{\eta}$ by a smooth manifold: $p: Z \to M$.

Let $Z = \{Z \in U \text{ s.t. } p^{-1}(Z) = \emptyset\}$. Extend $p: \hat{\eta}$ to constructible distribution $\mu$ on $M$.

By the induction assumption, $\mu$ is WF-holonomic.

Thus $p: \hat{\mu}$ is constructible WF-holonomic.

By the induction assumption $p: \hat{\mu}$ is WF-holonomic.
Idea of the proof.

- $\xi|_U$ is smooth for open dense $U$.
- Extend $\xi|_U$ to an holonomic constructible distribution $\xi'$.
- Let $\eta = \xi' - \xi$. We have $\dim \text{supp}(\eta) < \dim V$. 
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$$p : M \to Z$$
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- Let $Z' \subset Z$ open dense s.t. $p^{-1}(Z') \cong Z'$.
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- Let $Z' \subset Z$ open dense s.t. $p^{-1}(Z') \cong Z'$.
- Extend $p^*(\eta|_{Z'})$ to constructible distribution $\mu$ on $M$.  

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Idea of the proof.

- $\xi|_U$ is smooth for open dense $U$.
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- Let $\eta = \xi' - \xi$. We have $\dim \text{supp}(\eta) < \dim V$.
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- Let $Z' \subset Z$ open dense s.t. $p^{-1}(Z') \cong Z'$.
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- By the induction assumption, $\mu$ is WF-holonomic.
Idea of the proof.

- \( \xi|_U \) is smooth for open dense \( U \).
- Extend \( \xi|_U \) to an holonomic constructible distribution \( \xi' \).
- Let \( \eta = \xi' - \xi \). We have \( \dim \text{supp}(\eta) < \dim V \).
- Resolve \( Z = \text{supp}(\eta) \) by a smooth manifold:
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p : M \to Z
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- Let \( Z' \subset Z \) open dense s.t. \( p^{-1}(Z') \cong Z' \).
- Extend \( p^*(\eta|_{Z'}) \) to constructible distribution \( \mu \) on \( M \).
- By the induction assumption, \( \mu \) is WF-holonomic.
- Thus \( p_*(\mu) \) is constructible WF-holonomic.
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- Extend $\xi|_U$ to an holonomic constructible distribution $\xi'$.
- Let $\eta = \xi' - \xi$. We have $\dim \text{supp}(\eta) < \dim V$.
- Resolve $Z = \text{supp}(\eta)$ by a smooth manifold:

$$p : M \rightarrow Z$$

- $Z' \subset Z$ open dense s.t. $p^{-1}(Z') \cong Z'$.
- Extend $p^* (\eta|_{Z'})$ to constructible distribution $\mu$ on $M$.
- By the induction assumption, $\mu$ is WF-holonomic.
- Thus $p_*(\mu)$ is constructible WF-holonomic.
- By the induction assumption $p_*(\mu) - \eta$ is WF-holonomic.
The Key Lemma

**Key Lemma**

Let $f$ be a constructible function on an open (definable) set $U \subset V$. Then $f$ can be extended to a constructible WF-holonomic distribution on $V$. 

Idea of the Proof.

WLOG we can assume that the function $f$ has the form:

$$
\psi \hat{p}_1 \ast S \hat{p}_2 S \hat{p}_3 \ast 
$$

Using resolution of singularities we may assume that $U$ is the complement of the coordinate hyper planes and $p_i u_i m_i$, where $u_i$ are units and $m_i$ are monomials. While $u_2$ and $u_3$ can be ignored, $u_1$ cannot. Instead we can swallow it in $m_1$.

Now we prove the Key lemma for the complement of the origin. We are using an inductive assumption both about the Key lemma and the main theorem. Adding 1 point does not affect WF-holonomicity.
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- WLOG we can assume that the function $f$ has the form:
\[ \psi(p_1)|p_2|val(p_3) \]
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