$\mathfrak{z}$-finite distributions on p-adic groups

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$\mathcal{B}, n_i, W_i$ are explicitly described in terms of cuspidal representations of Levi subgroups of G.

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Theorem (Sakellaridis-Venkatesh, Delorme)

Many spherical pairs (including all symmetric pairs) satisfy: $\dim(\pi^*)^H < \infty$

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The theorem is reduced to the baby model using the theory of Bernstein center.

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- One can extend this definition to (analytic) manifolds.

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Theorem (Sayag)

Let π be an admissible representation of G. Then there are only finitely many non-nilpotent fuzzy balls s.t. $\pi(e_b) \neq 0$.

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  $0 = exp^*((e_B * \xi * e_B)|_U) = exp^*(e_B) * exp^*(\xi|_U) * exp^*(e_B) = exp^*(e_B) * exp^*(\xi|_U) = \widehat{1_B} * exp^*(\xi|_U) = \mathcal{F}(1_B \cdot \widehat{exp^*(\xi|_U)})$

Proof of the condition on the wave front set

- $\mathcal{S}(G) * \xi * \mathcal{S}(G)$ is admissible.
- $e_b * \mathcal{S}(G) * \xi * \mathcal{S}(G) * e_c = 0$ for many fuzzy balls b, c.
- $e_b * \xi * e_c = 0$ for many fuzzy balls b, c.
- $(\sum_{b \in X} e_b) * \xi * (\sum_{b \in X} e_b) = 0$ for many sets of fuzzy balls.
- $e_B * \xi * e_B = 0$ for many large balls B.
- $exp^*(e_B * \xi * e_B) = 0$ for many large balls B.
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- $exp^*(e_B * \xi * e_B) = 0$ for many large balls B.
- For small enough neighborhood U of 1:
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- $WF_0(exp^*(\xi|_U)) \subset \mathcal{N}$
- $WF_1(\xi) \subset \mathcal{N}$
- $WF_g(\xi) \subset \mathcal{N}$