Gelfand Pairs

A pair of compact topological groups \( G \supset H \) is called a **Gelfand pair** if the following equivalent conditions hold:

- \( L^1(G/H) \) decomposes to direct sum of distinct irreducible representations of \( G \).
- for any irreducible representation \( \pi \) of \( G \), \( dim \pi \leq 1 \).
- for any irreducible representation \( \pi \) of \( G \), \( dim \text{Hom}_{\pi}(\rho,C) \leq 1 \).
- the algebra of \( \pi \)-invariant functions on \( G \), \( C^\ast(G/H) \), is commutative w.r.t. convolution.

A pair of compact topological groups \( G \supset H \) is called a **strong Gelfand pair** if the following equivalent conditions hold:

- the pair \( G \supset H \) is a Gelfand pair.
- for any irreducible representations \( \pi \) of \( G \) and \( \rho \) of \( H \), \( \text{dim} \text{Hom}_{\pi}(\rho,C) \leq 1 \).
- the algebra of \( \pi \)-invariant functions on \( G \), \( C^\ast(G/H) \), is commutative w.r.t. convolution.

An analogous criterion works for strong Gelfand pairs.

**Gelfand Trick**

Let \( \theta \) be an involutive anti-automorphism of \( G \) (i.e. \( \theta(g)\theta(g) = g \)) and assume \( \theta(H) = H \). Suppose that \( \theta(f) = f \) for all \( \pi \)-invariant functions \( f \) on \( G \). Then \( (G,H) \) is a Gelfand pair.

**Tools to Work with Invariant Distributions**

Integration of distributions – Frobenius Decomposition

\[ \mathcal{D} \text{ modules well represent } \mathcal{D}^\ast \]

**Symmetric Pairs**

- A pair \( (G,H) \) is called a symmetric pair if \( H = G' \) for some involution \( \theta \).
- We denote \( \theta(g) = g^{-1} \).

**Symmetric Pairs**

**Gelfand Pairs**

A pair of groups \( (G,H) \) is called a Gelfand pair if for any irreducible (admissible) representation \( \rho \) of \( G \),

\[ \dim \text{Hom}_{\nu}(\rho,C) \leq 1. \]

For most pairs, this implies that \( \dim \text{Hom}_{\nu}(\rho,C) \leq 1 \).

**Gelfand-Kazhdan Distributional Criterion**

Let \( \rho \) be an involutive anti-automorphism of \( G \) and assume \( \rho(H) = H \).

Suppose that \( \rho(\xi) = \xi \) for all \( \pi \)-invariant distributions \( \xi \) on \( G \).

Then \( (G,H) \) is a Gelfand pair.

An analogous criterion works for strong Gelfand pairs.

**Results**

**Strong Gelfand Pairs**

Let \( (G,H) \) be a Gelfand pair.

- If \( G \) is compact:
  - \( \pi \) preserves closed \( \pi \)-invariant distributions on \( H \).

- If \( G \) is noncompact:
  - \( \pi \) preserves closed \( \pi \)-invariant distributions on \( H \).

\[ \sigma(x,y) = (Ax,y) \]

\[ \mathcal{X}(x,y) = (Ax,y) \]

**Example**

Any \( \pi \)-invariant distribution on the plane \( \mathbb{R}^2 \) is invariant with respect to the flip \( e \).

This example implies that \( (GL_n, GL_n) \) is a strong Gelfand pair.

**Tools to Work with Invariant Distributions**

Analysis

Integration of distributions – Frobenius Decomposition

Geometric Invariant Theory

Luna Slice Theorem

**Symmetric Pairs**

We call the property (2) regularity. We conjecture that all symmetric pairs are regular. This will imply the conjecture that every good symmetric pair is a Gelfand pair.

**Regular pairs**

<table>
<thead>
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<th>real case</th>
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