Gelfand Pairs

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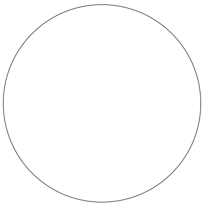
the compact case

Fourier Series

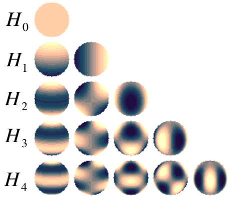


$$L^2(S^1) = \bigoplus_m Span(\chi_m)$$

$$\chi_m(t) = e^{imt}$$

$$\pi(\alpha)\chi_m = e^{im\alpha}\chi_m$$

Spherical Harmonics



$$S^2 = O_3 / O_2$$

$$L^2(S^2) = \bigoplus_m H_m$$

$$H_m = Span(Y_n^i)$$

are irreducible representations of O_3

Gelfand Pairs

A pair of compact topological groups $G \supset H$ is called a **Gelfand pair** if the following equivalent conditions hold:

- $L^2(G/H)$ decomposes to direct sum of **distinct** irreducible representations of G.
- for any irreducible representation ρ of G, $\dim \rho^H \leq 1$
- for any irreducible representation ρ of G, $\dim \mathrm{Hom}(\rho|_H, \mathbb{C}) \leq 1$
- the algebra of bi-H-invariant functions on G, $C(H \backslash G / H)$ is commutative w.r.t. convolution.

Strong Gelfand Pairs

A pair of compact topological groups $(G \supset H)$ is called a **strong Gelfand pair** if the following equivalent conditions hold:

- the pair $(G \times H \supset \Delta H)$ is a Gelfand pair.
- for any irreducible representations ρ of G and τ of H, $\dim \mathrm{Hom}(\rho|_H, \tau) \leq 1$.
- the algebra of $\mathrm{Ad}(H)$ - invariant functions on G is commutative w.r.t. convolution.

Gelfand Trick

Let σ be an involutive anti-automorphism of G (i.e. $\sigma(g_1 g_2) = \sigma(g_2)\sigma(g_1)$ and $\sigma^2 = \mathrm{Id}$) and assume $\sigma(H) = H$.

Suppose that $\sigma(f) = f$ for all bi-H-invariant functions $f \in C(H \backslash G / H)$. Then (G, H) is a Gelfand pair.

An analogous criterion works for strong Gelfand pairs.

Rep. theory: $\forall \rho \dim \rho^H \leq 1$ ⟺ Algebra: $C(H \backslash G/H)$ is commutative ⟸ "Analysis": $\exists$ anti-involution σ s.t. $f = \sigma(f)$ $\forall f \in C(H \backslash G/H)$ ⟺ Geometry: $\exists$ anti-involution σ that preserves H double cosets

Classical Examples

Pair	Anti-involution
$(G \times G, \Delta G)$	$(g,h) \mapsto (h^{-1}, g^{-1})$
$(O(n+k), O(n) \times O(k))$	
$(U(n+k), U(n) \times U(k))$	$g \mapsto g^{-1}$
$(GL(n,\mathbb{R}), O(n))$	$g \mapsto g^t$
(G, G^θ), where G - Lie group, θ- involution, G^θ is compact	$g \mapsto \theta(g^{-1})$
(G, K), where G - is a reductive group, K - maximal compact subgroup	Cartan anti-involution

Classical Applications

Gelfand-Zeitlin basis:

(S_n, S_{n-1}) is a strong Gelfand pair →

basis for irreducible representations of S_n. The same for $O(n,\mathbb{R})$ and $U(n,\mathbb{R})$.

Classification of representations:

$(GL(n,\mathbb{R}), O(n,\mathbb{R}))$ is a Gelfand pair →

the irreducible representations of $GL(n,\mathbb{R})$ which have an $O(n,\mathbb{R})$ - invariant vector are the same as characters of the algebra

$C(O(n,\mathbb{R}) \backslash GL(n,\mathbb{R}) / O(n,\mathbb{R}))$.

The same for the pair $(GL(n, \mathbb{C}), U(n))$.

the non compact case

In the non compact case we consider complex smooth (admissible) representations of algebraic reductive (e.g. GL_n, O_n, Sp_n) groups over local fields (e.g. $\mathbb{R}$, $\mathbb{Q}_p$).

Gelfand Pairs

A pair of groups $(G \supset H)$ is called a **Gelfand pair** if for any irreducible (admissible) representation ρ of G

$$\dim Hom_H(\rho, \mathbb{C}) \cdot \dim Hom_H(\tilde{\rho}, \mathbb{C}) \leq 1.$$

For most pairs, this implies that

$$\dim Hom_H(\rho, \mathbb{C}) \leq 1.$$

Gelfand-Kazhdan Distributional Criterion

Let σ be an involutive anti-automorphism of G and assume $\sigma(H) = H$.

Suppose that $\sigma(\xi) = \xi$ for all bi H- invariant distributions ξ on G.

Then (G, H) is a Gelfand pair.

An analogous criterion works for strong Gelfand pairs

Rep. theory: $\dim Hom_H(\rho, \mathbb{C}) \leq 1 \ \forall \rho$ ⟸ ~~Algebra~~ ⟸ Analysis: $\exists\ \sigma$ s.t. $\xi = \sigma(\xi)$ $\forall \xi \in \mathcal{S}^*(G)^{H \times H}$

Analysis ⇑ $p-adic$ Geometry: $\exists\ \sigma$ that preserves H double cosets ⟹ Geometry: $\exists\ \sigma$ that preserves closed H double cosets ⟹ Analysis

Results

Gelfand pairs

Pair	p-adic case	real case
$(GL_n(E), GL_n(F))$	Flicker	Aizenbud-Gourevitch
$(GL_{n+k}, GL_n \times GL_k)$	Jacquet-Rallis	
$(O_{n+k}, O_n \times O_k)$ over $\mathbb{C}$	————	
(GL_n, O_n) over $\mathbb{C}$		
(GL_{2n}, Sp_{2n})	Heumos-Rallis	Aizenbud-Sayag
$(GL_{2n}, \left\{\begin{pmatrix} g & u \\ 0 & g \end{pmatrix}\right\}, \psi)$	Jacquet-Rallis	Aizenbud-Gourevitch-Jacquet

Strong Gelfand pairs

Pair	p-adic case	real case
(GL_{n+1}, GL_n)	Aizenbud-Gourevitch-Rallis-Schiffmann	Aizenbud-Gourevitch Sun-Zhu
$(O(V \oplus F), O(V))$		Sun-Zhu
$(U(V \oplus F), U(V))$		

Example

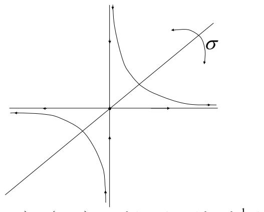
Any $F^\times$- invariant distribution on the plain F^2 is invariant with respect to the flip σ.

This example implies that **(GL_2, GL_1) is a strong Gelfand pair**.

More generally,

Any distribution on GL_{n+1} which is invariant w.r.t. conjugation by GL_n is invariant w.r.t. transposition.

This implies that **(GL_{n+1}, GL_n) is a strong Gelfand pair**.

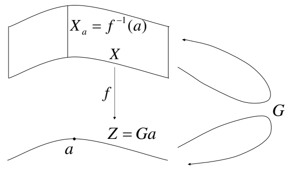


$$\sigma(x,y) = (y,x) \qquad \lambda(x,y) = (\lambda x, \lambda^{-1} y)$$

Tools to Work with Invariant Distributions

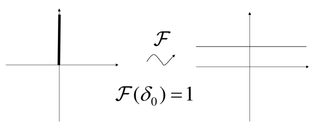
Analysis

Integration of distributions – Frobenius Descent



$$\mathcal{S}^*(X)^G = \mathcal{S}^*(X_a)^{G_a}$$

Fourier transform – uncertainty principle



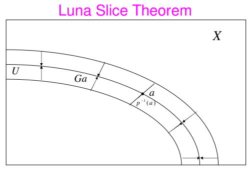
$$\mathcal{F}(\delta_0) = 1$$

Wave front set

Geometry

Geometric Invariant Theory

Luna Slice Theorem



Algebra

D – modules

Weil representation

Representations of SL_2

Symmetric Pairs

- A pair $(G \supset H)$ is called a **symmetric pair** if $H = G^\theta$ for some involution θ.
- We de\note $\sigma(g) := \theta(g^{-1})$.

Question: What symmetric pairs are Gelfand pairs?

- We call a symmetric pair (G, H, θ) **good** if σ preserves all closed H double cosets. Any connected symmetric pair over $\mathbb{C}$ is good.

Conjecture: Any good symmetric pair is a Gelfand pair.

Conjecture: Any symmetric pair over $\mathbb{C}$ is a Gelfand pair.

How to check that a symmetric pair is a Gelfand pair?

1. Prove that it is good
2. Prove that any H-invariant distribution on $\mathfrak{g}^\sigma$ is σ -invariant provided that this holds outside the cone of nilpotent elements.
3. Compute all the "descendants" of the pair and prove (2) for them.

We call the property (2) regularity. We conjecture that all symmetric pairs are regular. This will imply the conjecture that every good symmetric pair is a Gelfand pair.

Regular pairs

Pair	p-adic case by	real case by
$(G \times G, \Delta G)$	Aizenbud-Gourevitch	Aizenbud-Gourevitch
$(GL_n(E), GL_n(F))$	Flicker	
$(GL_{n+k}, GL_n \times GL_k)$	Jacquet-Rallis	
$(O_{n+k}, O_n \times O_k)$	Aizenbud-Gourevitch	
(GL_n, O_n)		
(GL_{2n}, Sp_{2n})	Heumos - Rallis	Aizenbud-Sayag
(Sp_{2m}, GL_m)	Aizenbud	Sayag (based on work of Sekiguchi)
(E_6, Sp_8)		
$(E_6, SL_6 \times SL_2)$		
(E_7, SL_8)		
(E_8, SO_{16})		
$(F_4, Sp_6 \times SL_2)$		
$(G_2, SL_2 \times SL_2)$		