SOME REPRESENTATION THEORY EXERCISES

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Lecture 1

Exercise 1.6

1) $V \boxtimes W \in irr(G \times H) \Longleftrightarrow V \in irr(G)$ & $W \in irr(H)$

$\Longrightarrow$

Suppose $V \boxtimes W \in irr(G \times H)$

If V is not irreducible $\Longrightarrow \exists$ V' $\subseteq V$ is a sub-representation of G

$\Longrightarrow V' \boxtimes H$ is a sub-representation of $G \times H$, contradiction with $V \boxtimes W$ irreducible.

Similar for W

$\Longleftarrow$

Suppose $V \in irr(G)$ & $W \in irr(H)$

i) $V \boxtimes W$ is a representation of $G \times \{e\}$

ii) Every irreducible sub-representation U of $G \times \{e\}$ inside $V \boxtimes W$ is of the form $V \boxtimes s, s \in W$

Let $\{w_0, w_1, ..w_n\}$ be a basis of W. Every element inside $V \boxtimes W$, or in U will be of the form:

$u = v_0 \boxtimes w_0 + v_1 \boxtimes w_1 + ... + v_n \boxtimes w_n$

U is an irreducible representation of $G \times \{e\}$

V is an irreducible representation of G.

We define morphisms $\sigma_i : U \to V$, $\sigma_i(u) = v_i$

σ_i is a linear map, and moreover, is a representation morphism between U and V.

By Schur's lemma, σ_i is an isomorphism. Moreover, $\sigma_i = \lambda_i * \sigma$, in which σ is a fixed isomorphism, $\lambda_i \in \mathbb{C}$.

$\to \sigma_i(u) = \lambda_i * \sigma(u) \to v_i = \lambda_i * \sigma(u)$

$\to u = \sum_i v_i \boxtimes w_i = \sum_i (\lambda_i * \sigma(u)) \boxtimes w_i = \sigma(u) \boxtimes (\sum_i \lambda_i * w_i)$

Let $s = \sum_i \lambda_i * w_i$, we have for all u in U, u = $\sigma(u) \boxtimes s$

σ is an isomorphism between U and V, so it's surjective. $\rightarrow$U $= V \boxtimes s$

iii) Suppose U is an invariant subspace of $V \boxtimes W$. We need to show $U = V \boxtimes W$

Let $U' \subseteq U$ be a sub-representation of $G \times \{e\}$

By ii), U' is of the form $V \boxtimes s \rightarrow V \boxtimes s \subseteq U$

W is an irreducible representation of H, so H acts transitively on W.

$\rightarrow$the image of $V \boxtimes s$ under the action of $G \times H$ is $V \boxtimes W \rightarrow V \boxtimes W \subseteq U$. Done.

2) $V \boxtimes W \simeq V' \boxtimes W' \Longleftrightarrow V \simeq V'$ & $W \simeq W'$

$\Longrightarrow$

Using the previous problem, part ii), irreducible sub-representations of $G \times \{e\}$ are of the form $V \boxtimes s$ and $V' \boxtimes t$, and there is a representation isomorphism between them

This isomorphism gives a representation isomorphism between V and V'. Similar for W and W'

$\Longleftarrow$

There exist representation isomorphisms A: $V \rightarrow V'$ and B: $W \rightarrow W'$

$A \boxtimes B : (V, W) \rightarrow V' \boxtimes W' : (v, w) \rightarrow Av \boxtimes Bw$ is a bilinear map.

It induces a bilinear map between $V \boxtimes W \rightarrow V' \boxtimes W'$, which can be shown to be a representation morphism. Using the previous exercise, $V \boxtimes W$ and $V' \boxtimes W'$ are irreducible, so the morphism is an isomorphism.

Exercise 1.10

1) $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y]) = \mathbb{C}(X \times Y)$

$X = x_1, x_2, ..., x_n$

$Y = y_1, y_2, .., y_m$

Define a map $\phi : \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y]) \rightarrow \mathbb{C}(X \times Y)$

Let f $\in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y])$; $f(x_i) = \sum_{j=1}^{m} c_{ij} * y_j, c_{ij} \in \mathbb{C}$

Define $\phi(f) = \sum_{i,j} c_{ij} * (x_i, y_j)$, which is in $C(X \times Y)$

ϕ is injective because if $\phi(f) = 0 \Rightarrow c_{ij} = 0 \Rightarrow f = 0$

ϕ is surjective because f is determined by $f(x_i)$, and we can choose $f(x_i)$ to be anything in $\mathbb{C}[Y]$, hence we can choose c_{ij} arbitrary.

2) $\mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y]) = \mathbb{C}(X \times Y)^G = \mathbb{C}(X \times Y/G)$

i) $\mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y]) = \mathbb{C}(X \times Y)^G$

Let π be the representation of C[X], and τ be the representation of C[Y]

$\mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y]) = \{f \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y]) \mid \tau(g)f\pi(g^{-1}) = f$ for all g in G$\}$

Let $f(x_i) = \sum\limits_j c_{ij} * y_j$ in $\mathrm{Hom}_G[\mathbb{C}[X], \mathbb{C}[Y]]$

Let $x_s = \pi(g^{-1})(x_i)$

$\to \tau(g)f\pi(g^{-1})(x_i) = \tau(g)f(x_s) = \tau(g)(\sum\limits_j c_{sj} * y_j) = \sum\limits_j c_{sj} * \tau(g)(y_j)$

$\to \sum\limits_j c_{ij} * y_j = \sum\limits_t c_{st} * \tau(g)(y_t)$

$\to$ if $y_j = \tau(g)(y_t)$ then $c_{ij} = c_{st}$

Conclusion: f satisfies $c_{ij} = c_{st}$ iff $x_i = \pi(g)x_s$ and $y_j = \tau(g)(y_t)$

$\sum\limits_{ij} c_{ij} * (x_i, y_j) \in C[X \times Y]^G$ iff $\sum\limits_{ij} c_{ij} * (x_i, y_j) = \sum\limits_{ij} c_{ij} * (\pi(g)(x_i), \tau(g)(y_j))$, and it is the same as the conclusion above, so we are done

ii) $\mathbb{C}(X \times Y)^G = \mathbb{C}(X \times Y/G)$:

$\mathbb{C}(X \times Y)^G = \{\sum c_{ij}(x_i, y_j) \mid \forall g \in G : \sum c_{ij}(x_i, y_j) = \sum c_{ij}(\pi(g)x_i, \tau(g)y_j)\}$

$\to \mathbb{C}(X \times Y)^G = \{\sum c_{ij}(x_i, y_j)$, in which all (x_i, y_j) in the same orbit have the same coefficient$\}$

From that, we are done.

3) $\dim \mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y]) = \#(X \times Y/G)$

Using 2), we have $\mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y]) = C(X \times Y/G)$

Take dim of both sides. Moreover, $\mathbb{C}(X \times Y/G)$ has dimension equal to the number of orbits under the action of G.

4) G is abelian, $irr(G) = ?$

Use the result: commutative operators share a common eigenvector (application of Hilbert's Nullstellensatz)

G is finite, and abelian, so $G = \oplus_{n_i}(\mathbb{Z}/n_i\mathbb{Z})$

The irreducible representation of a finite cyclic group $\mathbb{Z}/n_i\mathbb{Z}$ has dimension 1 (because if v is an eigenvector of 1, then v is also the eigenvector of all elements in the group)

The set $\{\pi(1_{n_i})\}$ consists of commuting matrices because G is abelian. Hence, there exists a common eigenvector v. Then the vector space generated by v is an invariant subspace of G. $\rightarrow$ irr(G) $= \mathbb{C}$

5) $Irr(S_4) = ?$

The number of irreducible representations of S_4 = the number of conjugacy classes = number of different cycle structures = 5

1) trivial representation, $\pi_1(g) = 1$

2) sign representation, $\pi_2(g) = sign(g)$

Let S_4 act on the set of vertices of a tetrahedron $X = \{x_1, x_2, x_3, x_4\}$

C[X] is a representation of S_4

C[X] has a sub-representation W of dimension 1 $= \{cx_1 + cx_2 + cx_3 + cx_4 \mid c \in \mathbb{C}\}$

$W^\perp = \{(c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \mid c_1 + c_2 + c_3 + c_4 = 0\}$

$\dim \mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[X]) = \#(X \times X/G) = 2$, because there are two orbits of $X \times X$ under the action of G: $\{(1,1), (2,2), (3,3), (4,4)\}$ and the rest.

It follows that $W^\perp$ is irreducible, because otherwise, we have at least 3 irreducible sub-representations of C[X], and $\dim \mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[X]) \geq 1 * 1 + 1 * 1 + 1 * 1 = 3$

So we have an irreducible representation of dimension 3:

3)$\pi_3 : W^\perp = \{(c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \mid c_1 + c_2 + c_3 + c_4 = 0\}$, of dimension 3

4)$\pi_3 \boxtimes \pi_2$ is an irreducible representation of G of dimension 3 ($W^\perp \boxtimes_{\mathbb{C}} \mathbb{C} \cong W^\perp$), and is non-isomorphic to π_3, because otherwise, using 2) of Exercise 1.6, we have the trivial and the sign representation be isomorphic

Let S_4 act on the set of edges of a tetrahedron $Y = \{x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}\}$

$\mathbb{C}[Y]$ is a representation of S_4

$\mathbb{C}[Y]$ has a sub-representation V of dimension 1 $= \{cx_{12} + cx_{13} + cx_{14} + cx_{23} + cx_{24} + cx_{34}\}$

$V^\perp = \{(c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34}) \mid (c_{12} + c_{13} + c_{14} + c_{23} + c_{24} + c_{34} = 0\}$, which is of dimension 5

The orbits of $Y \times Y$ are:

$\{(x_{12}, x_{12}), (x_{13}, x_{13}), (x_{14}, x_{14}), (x_{23}, x_{23}), (x_{24}, x_{24}), (x_{34}, x_{34})\}$

$\{(x_{12},x_{13}),(x_{13},x_{12}),(x_{12},x_{14}),(x_{14},x_{12}),(x_{13},x_{14}),(x_{14},x_{13}),$
$(x_{24},x_{34}),(x_{34},x_{24}),(x_{23},x_{34}),(x_{34},x_{23}),(x_{14},x_{24}),(x_{24},x_{14}),$
$(x_{12},x_{23}),(x_{23},x_{12}),(x_{13},x_{23}),(x_{23},x_{13}),(x_{12},x_{24}),(x_{24},x_{12}),$
$(x_{14},x_{34}),(x_{34},x_{14}),(x_{23},x_{24}),(x_{24},x_{23}),(x_{34},x_{13}),(x_{13},x_{34})\}$

$\{(x_{12},x_{34}),(x_{13},x_{24}),(x_{14},x_{23}),(x_{23},x_{14}),(x_{24},x_{13}),(x_{34},x_{12})\}$

So $\dim \mathrm{Hom}_G(\mathbb{C}[Y],\mathbb{C}[Y]) = 3 = 1^2+1^2+1^2$, so $\mathbb{C}[Y]$ is the direct sum of 3 non-isomorphic irreducible representations
$\mathbb{C}[Y]$ cannot contain an irreducible representation of dimension 4, because otherwise:
$\#S_4 = \sum_\rho \dim(\rho)^2 \geq 1^2 + 1^2 + 3^2 + 4^2 > 24$.
So $\mathbb{C}[Y]$ contains 1 representation of dim 1, 1 representation of dim 2, and 1 representation of dim 3. (6 = 1+ 2+3)
So we have already found 5 non-isomorphic irreducible representations: 2 representation of dimension one, 1 representation of dimension two, and 2 representations of dimension three.

To find an explicit formula for the representation of dimension 2 in C[Y] 1) we calculate $\mathrm{Hom}_G(\mathbb{C}[X],\mathbb{C}[Y])$
The set of orbits of $X \times Y$ is
$\{(x_1,x_{12}),(x_2,x_{12}),(x_1,x_{13}),(x_3,x_{13}),(x_1,x_{14}),(x_4,x_{14}),$
$(x_2,x_{23}),(x_3,x_{23}),(x_4,x_{24}),(x_2,x_{24}),(x_3,x_{34}),(x_4,x_{34})\}$

$\{(x_1,x_{23}),(x_1,x_{24}),(x_1,x_{34}),(x_2,x_{13}),(x_2,x_{14}),(x_2,x_{34}),$
$(x_3,x_{12}),(x_3,x_{14}),(x_3,x_{24}),(x_4,x_{12}),(x_4,x_{13}),(x_4,x_{23})\}$
The first orbit gives rise to a map from $\mathbb{C}[X] \to \mathbb{C}[Y]$
$x_1 \to x_{12} + x_{13} + x_{14}$
$x_2 \to x_{12} + x_{23} + x_{24}$
$x_3 \to x_{13} + x_{23} + x_{34}$
$x_4 \to x_{14} + x_{24} + x_{34}$
The image of $\mathbb{C}[X]$ in $\mathbb{C}[Y]$ is a space of dimension 4, taking the complement of that space, we get a vector space of dimension 2, which is the representation we are looking for. In particular, after calculation, we get the vector space is generated by $e_1 = (x_{12} - x_{14} - x_{23} + x_{34})$ and $e_2 = (x_{13} - x_{14} - x_{23} + x_2 4)$

S_4 is generated by (1 2) and (1 2 3 4).

(1 2) maps e_1 and e_2 to $e_1 - e_2$ and $-e_2$

(1 2 3 4) maps e_1 and e_2 to $-e_1$ and $e_2 - e_1$

So it's the 2 dimensional representation.