Tirgul 4 - Appendix

We want to show the statement which we started doing in class,

Exercise. *Let* $p : \tilde{X} \to X$ *be a path connected covering space, and let* $\varphi \in G(\tilde{X})$ *be a deck transformation, i.e.* $p \circ \varphi = p$*, then* φ *is determined by its action on a single point.*

Proof. Assume there is $\psi \in G(\tilde{X})$ s.t. $\varphi(\tilde{x}) = \psi(\tilde{x})$, let η be a path from $\tilde{x}$ to $\varphi(x) = \psi(x)$ and let $\tilde{y}$ be any point in $\tilde{X}$. Now, take $\gamma : I \to \tilde{X}$ to be a path s.t. $\gamma(0) = \tilde{y}$ and $\gamma(1) = \tilde{x}$. Both the paths $\gamma * \eta * \varphi \circ \gamma^{-1}$ and $\gamma * \eta * \psi \circ \gamma^{-1}$ are defined as $\psi \circ \gamma^{-1}$ and $\varphi \circ \gamma^{-1}$ are paths starting at $\psi(\tilde{x}) = \varphi(\tilde{x})$ (and ending at $\psi(\tilde{y})$ and $\varphi(\tilde{y})$ respectively). Since ψ and φ are deck transformations, we get that,

$$p \circ (\gamma * \eta * \varphi \circ \gamma^{-1}) = p \circ (\gamma * \eta * \psi \circ \gamma^{-1}).$$

By the unique lifting property of covering spaces we then must have that the two paths are the same, and in particular have the same end points, that is $\varphi(\tilde{y}) = \psi(\tilde{y})$. $\square$

Note that in particular that means that if $\varphi(\tilde{x}) = \tilde{x}$ for some $\tilde{x} \in \tilde{X}$ then φ is the identity element in $G(\tilde{X})$.