Algebraic Topology - Exercise 11

Solve the following questions:

1. Deduce the Mayer-Vietoris long exact sequence from the axioms of homology.
2. Prove that if $i : X \to Y$ is an embedding of simplicial complexes (that is, i is injective and $i(x)$ is isomorphic to X as a simplicial complex) then $C_i \cong X/Y$.
3. Let $X = \bigcup X_k$ be defined as in Exercise 10, and define:

$$R = \coprod_{k=0}^{\infty} X_k \times I/(r_k, 1) \sim (i_k(r_k), 0)$$

   where $i_k : X_k \hookrightarrow X_{k+1}$ is the inclusion.

   (a) show that $p : R \to X$ is a weak homotopy equivalence, where $p(r,t) = r$.

   (b) Let X be a simplicial complex with its skeleta X_n, and set $R_n = \coprod_{k=0}^{n} X_k \times I/ \sim$. Prove that $h_i(R, R_n) = 0$ for $i < n$ for any homology theory (hint: use axion 5). Deduce that $h_i(X, X_n) = 0$ for $i < n$ and that $h_i(X) \cong h_i(X_n) = 0$ for $i < n$.

4. Finish computing the homologies of the closed surfaces, that is compute $h(P\# \ldots \#P)$ (the homologies of the connected sum of n projective planes).

   Extra:

5. Prove that if $i : X \to X \vee Y$ is the standard embedding, then $\tilde{C}_i \cong Y$. Here X,Y are pointed spaces and $\tilde{C}_i$ denote the reduced cone.
6. Prove 3 (b) using Axiom 5".