<u>Algebraic Topology - Exercise 2</u>

Solve the following questions, you can use the facts that $\pi_1(S^1) = \mathbb{Z}$ and that $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.

1. Show that:
   a) $\pi_0 \circ \Omega = \pi_1$.
   b) $\Omega(\Omega(X)) = \mathrm{Mor}(S^2, X)$. [Hint: use the adjunction from class]

2. (Hatcher Chapter 1.1, Ex. 5) Show that for a space X TFAE:
   a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.
   b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.
   c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

3. (Hatcher Chapter 1.1 Ex 16 revised)
   a) Let $i : A \hookrightarrow X$ be the inclusion, prove that if A is a retract of X then the induced homomorphism $i_* : \pi_1(A) \to \pi_1(X)$ is injective.
   b) Show that there are no retractions $r : X \to A$ in the following cases:

      i. $X = \mathbb{R}^3$ with any subspace A homeomorphic to S^1.
      ii. $X = S^1 \times D^2$ with $A = S^1 \times S^1$ its boundary torus.

4. Given a topological space X, define a topological monoid $\Omega'(X)$ which is homotopically equivalent to $\Omega(X)$ and a map that sends the product of $\Omega'(X)$ to the non-associative product on $\Omega(X)$.

   [Hint: how can we fix the non-associativity of loop concatenation?]

5. Show that the torus is covered by a plane and by the cylinder.

   <u>Can hand in with Exercise 3 a week after this exercise sheet is due:</u>

6. Find a cover of a genus $n + 1$ surface by a genus $nk + 1$ surface.

   <u>Extra:</u>

7. Find a cover of $S^2 \vee S^1$ which is homotopically equivalent to $S^2 \vee S^2 \vee S^1$.