Algebraic Topology - Exercise 3

Solve the following questions:

0. Find a cover of a genus $n+1$ surface by a genus $nk+1$ surface (If you didn't hand this one last week).

1. Describe the universal cover and the fundamental group of the following spaces (you can use the fact that $\pi_1(S^2)$ is trivial):
   (a) The torus.
   (b) The cylinder.
   (c) The projective plane.
   (d) A bouquet of n circles, $S^1 \vee S^1 \vee ... \vee S^1$ (glued at the same point).
   (e) A finite graph. (You can assume that the graph has a spanning tree and that you can quotient out by it, prove you get something which is homotopically equivalent to $S^1 \vee \cdots \vee S^1$.)

2. Let $X = S^2 \vee S^2 \vee S^2 \vee \ldots$ (spheres glued at a common point).
   a) Show that X is homotopically equivalent to $\ldots \vee D^1 \vee S^2 \vee D^1 \vee S^2 \vee D^1 \vee \ldots$
   (this time each sphere is glued to a disc, and every two adacent discs are glued at one boundary point).
   b) What is the universal cover of $S^2 \vee S^1$? [Hint: use what we said about X in class.]

3. a) Show that for two spaces X and Y we have $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.
   b) What is the universal cover of $X \times Y$ given that $\tilde{X}$ and $\tilde{Y}$ are the universal covering spaces of X and Y respectively.

4. Find a way to hang a picture with n nails such that the removal of each nail will make the picture fall. (Mathematically, show that you can find a non-trivial element x in $\pi_1(\mathbb{R}^2 \backslash \{\text{n points}\})$ such that for any inclusion i into $\mathbb{R}^2 \backslash \{\text{n-1 points}\}$, we will have that $i_*(x)$ is the constant loop.)

5. Prove the homotopy lifting property: Let $f_t : Y \to X$ be a homotopy and $\tilde{f}_0 : Y \to \tilde{X}$ a lift of f_0 to a cover of X, then there exists a unique lift $f_t : Y \to \tilde{X}$ such that the corresponding diagram commutes, i.e. $p \circ \tilde{f}_t = f_t$ where p is the covering map.

Can hand in with Exercise 4 a week after this exercise sheet is due:

6. Let X be path connected, and $\phi : \tilde{X} \to X$ a covering map, show that:
   (a) $\tilde{X}$ is path connected $\iff$ the deck transformation group acts transitively on the fibers.
   (b) $|(\phi^{-1}(x_1)| = |\phi^{-1}(x_2)|$ for any $x_1, x_2 \in X$.

Extra exercises:

7. Do Question #1 with additional spaces:
   (a) The Mobius strip.
   (b) A high genus surface.
   (c) $S^1 \vee S^1 \vee S^1 \vee \dots$.
8. Show that a covering map is always open.
9. Let G act transitively on X, then $|G_x| = |G_y|$ for any $x, y \in X$.
10. Given a set X with two invertible maps $a, b : X \to X$ construct a cover of $S^1 \vee S^1$ such that the fiber at x_0 (point we glued S^1's at) will be X and the deck transformations will be a and b.