Algebraic Topology - Exercise 6

Solve the following questions:

1. Let G be a discrete group and X a simply connected space. Show that if G acts continuously on X and for any $x \in X$ there exists an open $x \in U \subseteq X$ such that $gU \cap U = \varnothing$ for all $g \in G$ then $\pi_1(X/G) = G$.

2. Prove that if $f : X \to Y$ and $g : X \to Y$ are homotopic then $f_* = g_*$ as homomorphisms $f_*, g_* : \pi_n(X) \to \pi_n(X)$ for all n.

3. Prove that $\pi_k(S^n)$ is trivial for $n > k$:

   (a) Show that any continuous $f : S^k \to \mathbb{R}^{n+1}$ can be approximated by a smooth map.

   (b) Show that any continuous $f : S^k \to S^n$ can be approximated by a smooth map.

   (c) Prove that any continuous $f : S^k \to S^n$ is homotopic to a smooth map.

   (d) Prove that if $f : S^k \to S^n$ is smooth then it is not surjective.

   (e) Conclude that any smooth map from S^k to S^n is homotopic to a constant map.

4. In the lecture we said that $\pi_1(X)$ acts on $\pi_n(X)$, what is this action for $n = 1$?