ALGEBRAIC TOPOLOGY-LECTURE 3

recall: A covering space of a space X is a space $\widetilde{X}$ together with a map $\varphi : \widetilde{X} \to X$ satisfying the following condition: there exists an open cover $\{U_\alpha\}$ of X such that for each α, $\varphi^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U_α by . Equivalent definition in the catagorical language: There exists a unique homeomorphism μ such that the following diagram is commutative (p is the trivial projection):

$$\begin{array}{ccc} X \times D & \overset{\exists! \mu}{\longrightarrow} & \widetilde{X} \\ \downarrow p & & \downarrow \varphi \\ X & \overset{id}{\longrightarrow} & X \end{array}$$

The empty disjoint union is allowed, so p need not be surjective.

Definition. X is *locally connected* if $\forall x \in X$ and for every neighborhood U of x, there exists$V \subseteq U$ such that V is connected. More generaly, X is locally "something" if $\forall x \in X$ and for every neighborhood U of x, there exists$V \subseteq U$ such that V is "something".

X is *simply connected* iff $\pi_1(X) = \{0\}$.

Example. Take $F = \{x, sin(1/x)\}_{x \in (0,1]} \cup \{0 \times [-1,1]\}$ and take some path γ between $(0,0) \in F$ and $(1, sin(1)) \in F$, and set $G = F \cup \{\gamma\}$. Then G is path connected, but not locally path connected (the point $(0,0)$ is the problematic one).

Definition. An isomorphism between covering spaces $p_1 : \widetilde{X_1} \to X$ and $p_2 : \widetilde{X_2} \to X$ is a homeomorphism $f : \widetilde{X_1} \to \widetilde{X_2}$ such that $p_1 = p_2 f$. This condition means exactly that f preserves the covering space structures, taking $p_1^{-1}(x)$ to $p_2^{-1}(x)$ for each $x \in X$. The inverse f^{-1} is then also an isomorphism, and the composition of two isomorphisms is an isomorphism, so we have an equivalence relation.

Exercise. Prove the *homotopy lifting property*: given a covering space $p : \widetilde{X} \to X$, a homotopy $f_t : Y \to X$, and a map $\widetilde{f_0} : Y \to \widetilde{X}$ lifting f_0 , then there exists a unique homotopy $\widetilde{f_t} : Y \to \widetilde{X}$ of $\widetilde{f_0}$ that lifts f_t .

Theorem. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p^*(\widetilde{X}, \widetilde{x_0})$ to the covering space $(\widetilde{X}, \widetilde{x_0})$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

The proof for this classification theorem can be found in Hatcher.

Recall: Let $H < G$ (not necessarily normal). Then there exists an action of G on G/H. Also, given an action of G on X then $G/G_{x_0} \cong X$ (isomorphism of $G - sets$).

For a covering space $p : \widetilde{X} \to X$, a path γ in X has a unique lift $\widetilde{\gamma}$ starting at a given point of $p^{-1}(\gamma(0))$, so we obtain a well-defined map $L_\gamma : p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$ by sending the starting point $\widetilde{\gamma}(0)$ of each lift $\widetilde{\gamma}$ to its ending point $\widetilde{\gamma}(1)$.This map defines an action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$.

Exercise. The action of $\pi_1(X, x_0)$ is transitive iff $\widetilde{X}$ is path connected.

Remark. In the lecture we reffered to this group action by "Deck transformations" but the definition of Deck transformation is a little different (next lecture).

Theorem. *1)* $p^* : \pi_1(\widetilde{X}, \widetilde{x_0}) \longrightarrow \pi_1(X, x_0)$ *is an embedding.*

2) $\pi_1(\widetilde{X}, \widetilde{x_0}) \cong \pi_1(X, x_0)_{\widetilde{x_0}}$ *via* p^* *(The left term is the stabilizer of* $\widetilde{x_0}$ *under the action of* $\pi_1(X, x_0)$*).*

Proof. 1) An element of the kernel of p^* is represented by a loop $\widetilde{\gamma} : I \to \widetilde{X}$ such that $p \circ \widetilde{\gamma} \sim const$. So we have an homotopy $f_t : I \to X$ of $f_0 = p \circ \widetilde{\gamma}$ to the trivial loop γ_{const}. By the homotopy lifting property, there exists an homotopy $\widetilde{f_t} : I \longrightarrow \widetilde{X}$ from $\widetilde{\gamma}$ to the lift of γ_{const}. But a lift of a constant path has to be a constant path as well since the fiber of x_0 is a discreat set. so $\widetilde{\gamma} \sim const$ and therefore $kerp^*$is trivial.

2) Let $[\widetilde{\gamma}] \in \pi_1(\widetilde{X}, \widetilde{x_0})$. Then $p^*([\widetilde{\gamma}]) = [p \circ \widetilde{\gamma}]$ and notice that $p \circ \widetilde{\gamma}$ stabilizes $\widetilde{x_0}$ since its lifting at $\widetilde{x_0}$ is $\widetilde{\gamma}$ and its a loop. On the other direction, let $[\gamma] \in \pi_1(X, x_0)_{\widetilde{x_0}}$then the lifting of γ at $\widetilde{x_0}$ is a loop $\widetilde{\gamma}$. so $p^*([\widetilde{\gamma}]) = [\gamma]$. So we have a surjective monomorphism so its an isomorphism. □

Example. 1) take the cover $p : R \longrightarrow S^1$. We have the morphism $p^* : \{0\} \longrightarrow \mathbb{Z}$.

2) Take the cover $p : S^1 \longrightarrow S^1$ by $e^{i\theta} \longmapsto e^{2i\theta}$. This corresponds to $p^* : \mathbb{Z} \longrightarrow \mathbb{Z}$ by $m \longmapsto 2m$.

Theorem. *Let* $p : \widetilde{X} \longrightarrow X$ *a covering. TFAE:*

a) $\widetilde{X}$ *is simply connected.*

b) $\widetilde{X}$ *doesn't have non trivial covering.*

c) $\pi_1(X, x_0)$ *acts freely on* $\varphi^{-1}(x_0)$ *(trivial stabilizer).*

d) For any cover $p' : \widetilde{X'} \longrightarrow X$ *there is a unique continous map* $\mu : \widetilde{X} \longrightarrow \widetilde{X'}$ *such that* $p' \circ \mu = p$.

Remark. Im not sure, but this theorem might demands X to be path-connected, locally path-connected, and semilocally simply-connected.

Corollary. *There exists a unique cover (up to isomorphism of covers) $\widetilde{X}$ satisfy the above conditions, and it is called the universal cover.*

Proof. If there are 2 simply connected covers $\widetilde{X_1}$and $\widetilde{X_2}$, then by d) there exists μ_1,μ_2 such that $p_2 \circ \mu_1 = p_1$ and $p_1 \circ \mu_2 = p_2$ so we have $p_1 \circ \mu_2 \circ \mu_1 = p_1$. But also from $d)$ it follows that id is the only continous map $\mu : \widetilde{X_1} \longrightarrow \widetilde{X_1}$ such that $p_1 \circ \mu = p_1$ so $\mu_2 \circ \mu_1 = id$ and in the same way $\mu_1 \circ \mu_2 = id$ so $\mu_1$is homeomorphism that preserves the fibers so an isomorphism of the covers above. □

Exercise. 1)$\varphi : \widetilde{X} \longrightarrow X$ a cover, X path connected, then $\varphi^{-1}(x_1) = \varphi^{-1}(x_2)$ for any $x_1, x_2 \in X$.

2) A cover is an open map.

3) If G acts transitively on X, then $|G_x| = |G_x|$for any $x, y \in X$.

Now for the proof of the theorem:

Proof. $a) \Longrightarrow b)$: We have the 1-1 map $p^* : \pi_1(\widetilde{X'}) \longrightarrow \pi_1(\widetilde{X})$ and we gave an exercise that if $\widetilde{X}$ is path connected then $\pi_1(\widetilde{X}) = \{0\}$ acts transitively on the fibers, so the fibers has to be single points and $\widetilde{X'}$ is trivial.

$a) \Longleftrightarrow c)$ Assume $\widetilde{X}$ is simply connected. So $\pi_1(\widetilde{X}) = \{0\}$. and we showed that $\pi_1(\widetilde{X}) \cong \pi_1(X)_{\widetilde{x_0}} = \{0\}$ so $\pi_1(X)$ acts freely on the fibers. For the other direction, read backwards...

$d) \Longrightarrow a)$ Assume $\widetilde{X}$ have a non trivial covering, then $\widetilde{X}$ is not simply connected and we can choose $\widetilde{X'}$to be a simply connected cover. So $p' : \widetilde{X'} \longrightarrow \widetilde{X}$ a non trivial covering.

Claim: $\widetilde{X'}$is a cover of X (without assuming locally simply connectedness of X, a cover of a cover doesnt have to be a cover if the fiber of the first cover p_1 is infinite, since if we choose U a neighborhood of$x$ such that $p^{-1}(U)$ is trivial, $p^{-1}(U)$ has infinite copies and the triviality of p_2 on each copy can be on a smaller and smaller neighborhood of the fibers of x such that the intersections of this neighborhood will be a single point). In order to fix this, we use the assumption of the localy simply connectedness.

proof: We take a small neighborhood U of X such that U is simply connected and $p^{-1}(U) = \mu(U \times D_1)$ when μ is homeomorphism (this is possible since X is locally-semisimple). Now look at each $\mu(U \times \{d\})$, It is semisimple since its homeomorphic to U and from $a) \Longrightarrow b)$ we now that since $p'^{-1}(\mu(U \times \{d\}))$ is a cover of a semisimple $\mu(U \times \{d\})$, it has to be trivial, i.e $p'^{-1}(\mu(U \times \{d\})) \cong \mu(U \times \{d\})) \times D_2 \cong U \times D_1 \times D_2$and its true for every d, so we get that $p'^{-1} \circ p^{-1}(U) \cong U \times D_1 \times D_2$ and therefore $\widetilde{X'}$ is a cover of X .But now we use $d)$ to get a unique continous map $\xi : \widetilde{X} \longrightarrow \widetilde{X'}$ such that $p \circ p' \circ \xi = p$.

Now, as we will see, for a simply connected cover $\widetilde{X'}$ we have $G(\widetilde{X'},\widetilde{X}) \cong \pi_1(\widetilde{X})$ so we have a non-trivial isomorphism of covers $\rho : \widetilde{X'} \longrightarrow \widetilde{X'}$corresponds to some lifting of $[\gamma] \in \pi_1(\widetilde{X})$(Hatcher p.71). Now observe that $p \circ p' \circ \rho \circ \xi = p \circ p' \circ \xi = p$ so $\xi = \rho \circ \xi$ form the uniqueness, so ρ stabilizes some point $\xi(\widetilde{x})$ in the fiber of $\widetilde{x}$, but $[\gamma]$ acts freely from (c) so $[\gamma] = [const]$ and ρ is trivial. contradiction.

$a) \Longrightarrow d)$ $\widetilde{X}$ is simply connected. Let $\psi : (\widetilde{X'},\widetilde{x'}) \longrightarrow (X, x_0)$ a covering. We take $\gamma : I \longrightarrow \widetilde{X}$ from $\gamma(0) = \widetilde{x_0}$ to $\gamma(1) = \widetilde{x}$, and consider γ' to be the lifting by ψ of $p \circ \gamma$ at the point $\widetilde{x'}$ Define the following map $\tau : \widetilde{X} \longrightarrow \widetilde{X'}$ by $\tilde{x} \longmapsto \gamma'(1)$. So we get that $\psi \circ \tau = p$. Observe that:

*τ is well defined since if we take a different $\beta : I \longrightarrow \widetilde{X}$ from $\beta(0) = \widetilde{x_0}$ to $\beta(1) = \widetilde{x}$ then $\gamma * \beta^{-1} \sim id$ since $\widetilde{X}$ is simply connected. so $p^*([\gamma * \beta^{-1}]) = [const]$ and therefore the lifting by ψ will be in the kernel of ψ^* which is a loop at $\widetilde{x'}$. Also $p \circ (\gamma * \beta^{-1}) = p \circ \gamma * p \circ \beta^{-1}$ and $lift(p \circ \gamma * p \circ \beta^{-1}) = lift(p \circ \gamma) * lift(p \circ \beta^{-1})$. so we see that the end point of the lift of $p \circ \gamma$ is the same as the end point of the lift of $p \circ \beta$ and since $lift(p \circ \gamma * p \circ \beta^{-1})$ is a loop at $\widetilde{x'}$ then both $lift(p \circ \gamma)$ and $lift(p \circ \beta)$ starts at $\widetilde{x'}$so by definition, τ doesn't depend on the choise of β and γ.

* τ is continous: Take an open $U \subseteq \widetilde{X'}$ such that $\psi^{-1} \circ \psi(U) \cong U \times D$. Now observe that $p^{-1}\psi(U) = \tau^{-1} \circ \psi^{-1}\psi(U)$ and $p^{-1}\psi(U)$ is open, so $\tau^{-1} \circ \psi^{-1}\psi(U)$ is open. But $\psi^{-1}\psi(U)$ contains a disjoint union of U and othe sets homeomorphic to U so by the properties of disjoint union this implies that $\tau^{-1}(U)$ is open.

The last observation is that the open sets U such that $\psi^{-1} \circ \psi(U) \cong U \times D$ form a basis for the topology on $\widetilde{X'}$(since for any open set $V \subseteq \widetilde{X'}$ and any point $\widetilde{x} \in V$ we can take a neighborhood B in $\psi(V)$ of $\psi(\widetilde{x})$ such that $\psi^{-1}(B) \cong B \times D$ via μ. so $\widetilde{x} \in \mu^{-1}(B \times \{d\})$ for some $d \in D$ so $W = \mu^{-1}(B \times \{d\}) \cap V \subseteq V$ $\psi^{-1} \circ \psi(W) \cong W \times D$ and it is an element of the basis. So its indeed a basis. So τ is continous.

* τ is unique, since if we have also τ' then taking any $\gamma(0) = \widetilde{x_0}$and $\gamma(1) = \widetilde{x}$ then both $\tau \circ \gamma$, $\tau' \circ \gamma$ are two lifting of $p \circ \gamma$ and they have the same starting point since τ', τ are continous maps on pointed topological spaces so $\tau(\widetilde{x_0}) = \tau'(\widetilde{x_0}) = \widetilde{x'}$. So they have the same lift so $\tau \circ \gamma = \tau' \circ \gamma$ and $\tau(\widetilde{x}) = \tau'(\widetilde{x})$.

$b) \Longrightarrow a)$ $\widetilde{X}$ doesn't have non trivial covering, then $\widetilde{X}$ has to be simply connected, since every space Y (satisfying some local connectedness conditions) has a semisimple cover and if $\widetilde{X}$ is not simply connected then this semisimple cover will be non trivial. For the construction, see "tirgul 3" or hatcher. □