Decimal representation, Cantor set, p-adic numbers and Ostrowski Theorem.

A. Aizenbud

Weizmann Institute of Science

http://www.wisdom.weizmann.ac.il/~aizenr

Approaches to the real numbers

Approaches to the real numbers

- axiomatic approach

Approaches to the real numbers

- axiomatic approach
- Cauchy sequences

Approaches to the real numbers

- axiomatic approach
- Cauchy sequences
- Dedekind sections

Approaches to the real numbers

- axiomatic approach
- Cauchy sequences
- Dedekind sections
- decimal representations

Approaches to the real numbers

- axiomatic approach
- Cauchy sequences
- Dedekind sections
- decimal representations

The last is the most elementary but have some problems:

Approaches to the real numbers

- axiomatic approach
- Cauchy sequences
- Dedekind sections
- decimal representations

The last is the most elementary but have some problems:

- Hard to perform simple operations:

Approaches to the real numbers

- axiomatic approach
- Cauchy sequences
- Dedekind sections
- decimal representations

The last is the most elementary but have some problems:

- Hard to perform simple operations:

$$0.666\cdots + 0.33\cdots 33233\cdots = 0.\cdots$$

Approaches to the real numbers

- axiomatic approach
- Cauchy sequences
- Dedekind sections
- decimal representations

The last is the most elementary but have some problems:

- Hard to perform simple operations:

$$0.666\cdots + 0.33\cdots 33233\cdots = 0.\cdots$$

but

$$0.666\cdots + 0.33\cdots 33433\cdots = 1.\cdots$$

Approaches to the real numbers

- axiomatic approach
- Cauchy sequences
- Dedekind sections
- decimal representations

The last is the most elementary but have some problems:

- Hard to perform simple operations:

$$0.666\cdots + 0.33\cdots 33233\cdots = 0.\cdots$$

but

$$0.666\cdots + 0.33\cdots 33433\cdots = 1.\cdots$$

- Multiple representation:

- axiomatic approach
- Cauchy sequences
- Dedekind sections
- decimal representations

The last is the most elementary but have some problems:

- Hard to perform simple operations:

$$0.666\cdots + 0.33\cdots 33233\cdots = 0.\cdots$$

but

$$0.666\cdots + 0.33\cdots 33433\cdots = 1.\cdots$$

- Multiple representation:

$$0.9999\cdots = 1.000\cdots$$

Decimal representation

Decimal representation

Decimal representation can be viewed as a map

$$D : \{0, \dots, 9\}^{\mathbb{N}} \to [0, 1].$$

Decimal representation can be viewed as a map

$$D : \{0, \dots, 9\}^{\mathbb{N}} \to [0, 1].$$

This map is

Decimal representation

Decimal representation can be viewed as a map

$$D : \{0, \ldots, 9\}^{\mathbb{N}} \to [0, 1].$$

This map is

- Onto

Decimal representation

Decimal representation can be viewed as a map

$$D : \{0, \dots, 9\}^{\mathbb{N}} \to [0, 1].$$

This map is

- Onto
- Continuous

Decimal representation

Decimal representation can be viewed as a map

$$D : \{0, \ldots, 9\}^{\mathbb{N}} \to [0, 1].$$

This map is

- Onto
- Continuous
- A topological quotient

Decimal representation

Decimal representation can be viewed as a map

$$D : \{0, \ldots, 9\}^{\mathbb{N}} \to [0, 1].$$

This map is

- Onto
- Continuous
- A topological quotient

BUT NOT 1-1

Decimal representation

Decimal representation can be viewed as a map

$$D : \{0, \dots, 9\}^{\mathbb{N}} \to [0, 1].$$

This map is

- Onto
- Continuous
- A topological quotient

BUT NOT 1-1

If it would be 1-1 it would be an homeomorphism.

The Cantor set

The Cantor set

Exercise

$\{0, \ldots, 9\}^{\mathbb{N}} \cong$

Exercise

$\{0,\ldots,9\}^{\mathbb{N}} \cong \{0,\ldots,n\}^{\mathbb{N}} \cong$

The Cantor set

Exercise

$\{0,\ldots,9\}^{\mathbb{N}} \cong \{0,\ldots,n\}^{\mathbb{N}} \cong \prod F_i \cong$

Exercise

$$\{0,\ldots,9\}^{\mathbb{N}} \cong \{0,\ldots,n\}^{\mathbb{N}} \cong \prod F_i \cong \{(0.a_1\cdots)_3 | a_i \in \{0,1\}\}$$

The Cantor set

Exercise

$\{0,\dots,9\}^{\mathbb{N}} \cong \{0,\dots,n\}^{\mathbb{N}} \cong \prod F_i \cong \{(0.a_1\cdots)_3 | a_i \in \{0,1\}\} \cong$ *Cantor set*

The Cantor set

Exercise

$\{0,\ldots,9\}^{\mathbb{N}} \cong \{0,\ldots,n\}^{\mathbb{N}} \cong \prod F_i \cong \{(0.a_1\cdots)_3 | a_i \in \{0,1\}\} \cong$ *Cantor set*

p-adic numbers

p-adic numbers

Definition

$$\mathbb{Q}_p = \{(\cdots a_0.a_1 \cdots a_n) | a_i \in \{0, \ldots, p-1\}\}$$

p-adic numbers

Definition

$$\mathbb{Q}_p = \{(\cdots a_0.a_1 \cdots a_n) | a_i \in \{0, \ldots, p-1\}\}$$

- $+, -, \cdot$ are easy to define

p-adic numbers

Definition

$$\mathbb{Q}_p = \{(\cdots a_0.a_1 \cdots a_n) | a_i \in \{0, \ldots, p-1\}\}$$

- $+, -, \cdot$ are easy to define

Exercise

One can define ":" iff p is prime.

p-adic numbers

Definition

$$\mathbb{Q}_p = \{(\cdots a_0.a_1 \cdots a_n) | a_i \in \{0, \ldots, p-1\}\}$$

- $+, -, \cdot$ are easy to define

Exercise

One can define ":" iff p is prime.

Definition (p-adic absolute value)

$|p^n \cdot \frac{a}{b}|_p := p^{-n}$, where $(a, p) = (b, p) = 1$

Definition

$$\mathbb{Q}_p = \{(\cdots a_0.a_1 \cdots a_n) | a_i \in \{0, \ldots, p-1\}\}$$

- $+, -, \cdot$ are easy to define

Exercise

One can define ":" iff p is prime.

Definition (p-adic absolute value)

$|p^n \cdot \frac{a}{b}|_p := p^{-n}$, where $(a, p) = (b, p) = 1$

Exercise

Equivalent definition of $\mathbb{Q}_p$: Completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$

Absolute Value

Absolute Value

Definition (absolute value)

- absolute value on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ s.t.

Absolute Value

Definition (absolute value)

- absolute value on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ s.t.
  - $|a| = 0$ iff a=0,

Absolute Value

Definition (absolute value)

- absolute value on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ s.t.
  - $|a| = 0$ iff a=0,
  - $|ab| = |a||b|$,

Absolute Value

Definition (absolute value)

- absolute value on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ s.t.
  - $|a| = 0$ iff a=0,
  - $|ab| = |a||b|$,
  - $|a + b| \leq |a| + |b|$.

Absolute Value

Definition (absolute value)

- absolute value on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ s.t.
  - $|a| = 0$ iff a=0,
  - $|ab| = |a||b|$,
  - $|a + b| \leq |a| + |b|$.
- $|\cdot|_\alpha$ and $|\cdot|_\beta$ are equivalent iff they define the same topology (i.e. the same notion of convergence)

Definition (absolute value)

- absolute value on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ s.t.
  - $|a| = 0$ iff a=0,
  - $|ab| = |a||b|$,
  - $|a+b| \leq |a| + |b|$.
- $|\cdot|_\alpha$ and $|\cdot|_\beta$ are equivalent iff they define the same topology (i.e. the same notion of convergence)

Exercise

Absolute Value

Definition (absolute value)

- absolute value on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ s.t.
  - $|a| = 0$ iff a=0,
  - $|ab| = |a||b|$,
  - $|a+b| \leq |a| + |b|$.
- $|\cdot|_\alpha$ and $|\cdot|_\beta$ are equivalent iff they define the same topology (i.e. the same notion of convergence)

Exercise

- $|\cdot|_\alpha$ *and* $|\cdot|_\beta$ *are equivalent iff* $|\cdot|_\alpha^a = |\cdot|_\beta$

Absolute Value

Definition (absolute value)

- absolute value on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ s.t.
  - $|a| = 0$ iff a=0,
  - $|ab| = |a||b|$,
  - $|a+b| \leq |a| + |b|$.
- $|\cdot|_\alpha$ and $|\cdot|_\beta$ are equivalent iff they define the same topology (i.e. the same notion of convergence)

Exercise

- $|\cdot|_\alpha$ *and* $|\cdot|_\beta$ *are equivalent iff* $|\cdot|_\alpha^a = |\cdot|_\beta$
- *We have the following dichotomy*

Definition (absolute value)

- absolute value on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ s.t.
  - $|a| = 0$ iff a=0,
  - $|ab| = |a||b|$,
  - $|a+b| \leq |a| + |b|$.
- $|\cdot|_\alpha$ and $|\cdot|_\beta$ are equivalent iff they define the same topology (i.e. the same notion of convergence)

Exercise

- *$|\cdot|_\alpha$ and $|\cdot|_\beta$ are equivalent iff $|\cdot|_\alpha^a = |\cdot|_\beta$*
- *We have the following dichotomy*
  - *Either $|\cdot|$ is Archimedean, i.e. $|\mathbb{Z}|$ is unbounded*

Absolute Value

Definition (absolute value)

- absolute value on a field F is a map $|\cdot|: F \to \mathbb{R}_{\geq 0}$ s.t.
  - $|a| = 0$ iff a=0,
  - $|ab| = |a||b|$,
  - $|a+b| \leq |a| + |b|$.
- $|\cdot|_\alpha$ and $|\cdot|_\beta$ are equivalent iff they define the same topology (i.e. the same notion of convergence)

Exercise

- $|\cdot|_\alpha$ *and* $|\cdot|_\beta$ *are equivalent iff* $|\cdot|_\alpha^a = |\cdot|_\beta$
- *We have the following dichotomy*
  - *Either* $|\cdot|$ *is Archimedean, i.e.* $|\mathbb{Z}|$ *is unbounded*
  - *Or* $|\cdot|$ *is non-Archimedean, i.e.* $|a+b| \leq \max(|a|,|b|)$.

Absolute Value

Definition (absolute value)

- absolute value on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ s.t.
  - $|a| = 0$ iff a=0,
  - $|ab| = |a||b|$,
  - $|a + b| \leq |a| + |b|$.
- $|\cdot|_\alpha$ and $|\cdot|_\beta$ are equivalent iff they define the same topology (i.e. the same notion of convergence)

Exercise

- *$|\cdot|_\alpha$ and $|\cdot|_\beta$ are equivalent iff $|\cdot|_\alpha^a = |\cdot|_\beta$*
- *We have the following dichotomy*
  - *Either $|\cdot|$ is Archimedean, i.e. $|\mathbb{Z}|$ is unbounded*
  - *Or $|\cdot|$ is non-Archimedean, i.e. $|a + b| \leq \max(|a|, |b|)$.*
- *Show that if $|\cdot|$ is non-Archimedean then $|\cdot|^a$ is also a (non-Archimedean) absolute value*

Ostrowski Theorem

Ostrowski Theorem

Theorem (Ostrowski)

the possible (up to equivalence) absolute values on $\mathbb{Q}$ are:

Theorem (Ostrowski)

the possible (up to equivalence) absolute values on $\mathbb{Q}$ are:

- *Archimedean: the usual one* – $|\cdot|_\infty$.

Ostrowski Theorem

Theorem (Ostrowski)

the possible (up to equivalence) absolute values on $\mathbb{Q}$ are:

- *Archimedean: the usual one – $|\cdot|_\infty$.*
- *non-Archimedean: the p-adic ones – $|\cdot|_p$.*

Ostrowski Theorem

Theorem (Ostrowski)

the possible (up to equivalence) absolute values on $\mathbb{Q}$ are:

- *Archimedean: the usual one –* $|\cdot|_\infty$.
- *non-Archimedean: the p-adic ones –* $|\cdot|_p$.

Exercise

What powers of $|\cdot|_\infty$ *are absolute value*

Local Fields

Local Fields

Definition (Local Field)

A local field is a locally compact non-discrete topological field.

Definition (Local Field)

A local field is a locally compact non-discrete topological field.

Theorem (Classification of Local Fields)

The possible local fields are:

Definition (Local Field)

A local field is a locally compact non-discrete topological field.

Theorem (Classification of Local Fields)

The possible local fields are:

- *positive Characteristic:* $\mathbb{F}_q((t))$

Definition (Local Field)

A local field is a locally compact non-discrete topological field.

Theorem (Classification of Local Fields)

The possible local fields are:

- *positive Characteristic:* $\mathbb{F}_q((t))$
- *zero Characteristic:*

Local Fields

Definition (Local Field)

A local field is a locally compact non-discrete topological field.

Theorem (Classification of Local Fields)

The possible local fields are:

- *positive Characteristic:* $\mathbb{F}_q((t))$
- *zero Characteristic:*
  - *non-Archimedean: finite extortions of* $\mathbb{Q}_p$

Local Fields

Definition (Local Field)

A local field is a locally compact non-discrete topological field.

Theorem (Classification of Local Fields)

The possible local fields are:

- *positive Characteristic:* $\mathbb{F}_q((t))$
- *zero Characteristic:*
  - *non-Archimedean: finite extortions of* $\mathbb{Q}_p$
  - *Archimedean:*

Local Fields

Definition (Local Field)

A local field is a locally compact non-discrete topological field.

Theorem (Classification of Local Fields)

The possible local fields are:

- *positive Characteristic:* $\mathbb{F}_q((t))$
- *zero Characteristic:*
  - *non-Archimedean: finite extortions of* $\mathbb{Q}_p$
  - *Archimedean:*
    - $\mathbb{R}$

Definition (Local Field)

A local field is a locally compact non-discrete topological field.

Theorem (Classification of Local Fields)

The possible local fields are:

- *positive Characteristic:* $\mathbb{F}_q((t))$
- *zero Characteristic:*
  - *non-Archimedean: finite extortions of* $\mathbb{Q}_p$
  - *Archimedean:*
    - $\mathbb{R}$
    - $\mathbb{C}$